

Delta Plots and Coherent Distribution Ordering

Paul L. SPECKMAN, Jeffrey N. ROUDER, Richard D. MOREY, and Michael S. PRATTE

Social scientists often compare subclasses of populations or manipulations. For example, in comparing task-completion times across two levels of a manipulations, if one group has faster overall mean response, it is natural to ask if the fastest 10% of the first group has a faster mean than the fastest 10% of the second group, and so on. Delta plots, a type of quantile-quantile residual plot used by psychologists, shed light on these comparisons and motivate new notions of stochastic ordering. If all percentile classes have faster mean in one group than in the other, we say that there is coherent mean ordering and that one group stochastically dominates the other in mean. A related notion of coherent variance ordering can be defined similarly. Violations of coherent orderings of means or variances are diagnostic signatures of complex effects and suggest further avenues of study. In this note, we derive necessary and sufficient conditions for stochastic dominance in mean and variance. We show that stochastic mean dominance is exactly equivalent to the usual stochastic dominance and stochastic variance dominance is equivalent to ordering of the first derivative of the quantile functions.

KEY WORDS: Delta plot; Q-Q plot; Stochastic dominance.

Statisticians are familiar with quantile plots and Q-Q plots (e.g., Wilk and Gnanadesikan 1968). Psychologists have introduced a variant called the *delta plot* (De Jong 1994) that brings into sharp focus differences in populations on a quantile scale. The purpose of this note is to show how the delta plot can be used to gain insight into experimental data and how it relates to notions of stochastic ordering.

To define the delta plot, for simplicity we assume complete information on two distributions, X and Y , with respective quantile functions $Q_X(p)$ and $Q_Y(p)$. The population delta plot is a graph of $Q_Y(p) - Q_X(p)$ against $[Q_X(p) + Q_Y(p)]/2$. In practice, the delta plot is estimated from data using sample quantile functions. Another closely related plot is a plot of differences $Q_Y(p) - Q_X(p)$ against probability p . One advantage of plotting the difference in quantiles against the mean rather than p is that the delta plot for two distributions in a location-scale family is linear. This linearity property is also true for an ordinary Q-Q plot, but the Q-Q plot is not as useful for isolating

differences between two distributions.

There is considerable intuitive information in a delta plot. Figure 1 shows three examples from data collected in our experimental psychology laboratory. The line labeled "Stroop" is data from a classic task in psychology, the Stroop Task (Stroop 1935). In our case, participants were presented words in colored fonts and had to report the color by pressing an appropriate key. The words, however, were color terms such as "RED" and "GREEN." The task is easy if the font color matches the term; for example, if the word "RED" is presented in red. The task is quite difficult, however, if there is a mismatch; for example, if the word "RED" is presented in green. In this latter case, the meaning of the word interferes with the judgment of the font color. The former and latter conditions are called *congruent* and *incongruent*, respectively, and the time taken to make a judgment is called the response time (RT). The resulting delta plot is the line labeled "Stroop" (see note below). Let X and Y denote response times to the congruent and incongruent conditions, respectively. There is a stochastic ordering— X is said to be stochastically smaller than Y if $F_X(t) \geq F_Y(t)$ for all t . An immediate consequence of stochastic ordering is that the quantile functions are also ordered: $Q_Y(p) \geq Q_X(p)$ for $0 < p < 1$. Thus, Y stochastically dominates X if and only if the delta plot is nonnegative.

Figure 1 also shows data from a second task, the Simon Task (Simon 1969). [Note: A group of 38 participants observed 504 trials in the Stroop task followed by 504 trials in the Simon task. Sample quantiles were estimated for each participant in each condition in each task and averaged across participants. A separate group of 37 participants observed 84 trials in the phone-number repetition Simon task.] The task is similar; the participant identifies the color of boxes, either red or green, by pressing a key on the left for red and on the right for green. These colored boxes are presented on either the left or right side of the display. In the congruent condition, the side of display matches the correct answer; for example, a red square presented on the left. In the incongruent condition, there is a mismatch between the side of response and the side of presentation. The resulting delta plots for the comparison of congruent and incongruent conditions are shown with the line labeled "SIMON." In this case, there appears to be no consistent stochastic ordering of the distributions since the delta plot is both positive and negative.

Assuming these features of the delta plots are real and not just random fluctuations, they suggest a few substantive directions for exploration: Although the Simon and Stroop tasks are similar, they may be mediated by different cognitive mechanisms. The apparent lack of stochastic ordering for the Simon interference is a diagnostic clue to the underlying psychology. The effect of congruency is to speed only the fast responses while slowing the slower ones; this divergence suggests two distinct mental processes. Ridderinkopf (2002), for instance, proposed a theory in which the initial reaction to a color block is a quick,

Paul L. Speckman is Professor of Statistics, University of Missouri; Jeffrey Roudier is Associate Professor of Psychology, University of Missouri; Richard Morey is Assistant Professor of Psychology, University of Groningen, Netherlands; and Michael Pratte is graduate student in psychology, University of Missouri. Correspondence may be directed to Paul L. Speckman, Department of Statistics, Columbia, MO 65211 (E-mail: speckmanp@missouri.edu). This research was supported in part by NSF grants SES-0095919 and SES-0351523 and NIMH grant R01-MH071418. The authors are grateful to the Editor and an Associate Editor for stimulating comments and suggestions.

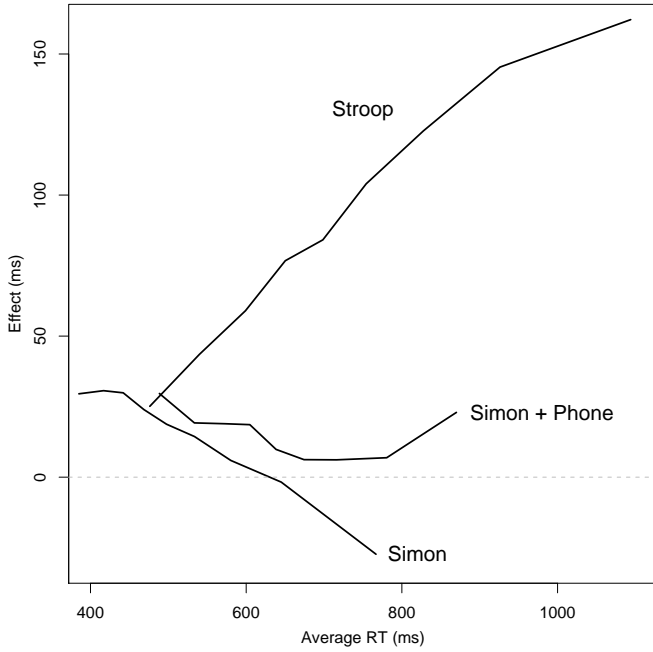


Figure 1. Delta plots for three conditions: Stroop interference, Simon interference, and Simon interference with phone-number repetition.

automatic activation of the response on the side of presentation. Following this reaction, there may be a slower compensating tendency that relatively favors the opposite side (see also Huber et al. 2002).

The final delta-plot line, labeled “Simon+Phone,” shows the results of a modified Simon task (see previous note). In this task, participants were asked to repeat over-and-over their phone number during the trial. The resulting delta plots suggest fairly complex dynamics that are concordant with a mixture model. Consider the following speculation: For the majority of responses, the repetition of the phone number proceeded smoothly with no effect on Simon interference whatsoever. For a minority of responses, however, the phone number repetition affected processing, perhaps through a misarticulation of the phone number. In this case, the response to the square was slowed, and the compensatory response was not engaged. The observed delta plot is concordant with such a mixture.

The two-process suggestion from the Simon delta plot came about because the mean of the fast responses, say the fastest 20% in each distribution, is smaller for the congruent condition. The mean of the slowest responses, say the slowest 20%, is smaller for the incongruent condition. In a sense to be made more precise below, the position of the delta plot provides for comparison of means *at percentile levels*. The mixture suggestion from the Simon+Phone delta plot came about because the slopes reverse. The slope of the delta plot provides for a comparison of variance *at percentile levels*. The fastest 20% is more variable in the congruent condition than the incongruent; the slowest 20% is more variable for the incongruent than the congruent. When conditional means or conditional variances reverse orderings across a distribution, it is surely a sign of complex dynamics that warrant further investigation as diagnostic phenomena.

1. ORDERING DISTRIBUTIONS

Recall that ranking distributions by their unconditional means is equivalent to first-order stochastic dominance (FOSD) under strong conditions, for example, in location parameter families such as normal distributions with equal variances. A delta plot that is everywhere positive shows that the distributions may be ranked by their means. However, the delta plot suggests more. In comparing segments of the distributions such as the fastest 10% of each, a positive delta plot would seem to imply that FOSD implies dominance of the conditional means. That is, the mean of the fastest 10% in one group is greater than the mean of the fastest 10% in the other group, and so on. Similarly, a positive slope to the delta plot suggests that the variance of the plot for the fastest 10% would seem to imply that the conditional variance of the fastest 10% in one group is greater than the other, etc. We can make these observations more precise.

We define concepts of *coherent mean ordering* (CMO) and *coherent variance ordering* (CVO) as follows. In the notation above, let X and Y denote two absolutely continuous random variables with common support, CDFs F_X and F_Y , and quantile functions $Q_X(p) = F_X^{-1}(p)$ and $Q_Y(p) = F_Y^{-1}(p)$ for $0 < p < 1$. Finally, for $0 \leq p < q \leq 1$, let X_{pq} denote the truncated random variable with distribution $(X | Q_X(p) \leq X \leq Q_X(q))$, that is, with cdf $F_{X_{pq}}(x) = (F_X(x) - p)/(q - p)$ for $Q_X(p) \leq x \leq Q_X(q)$. The truncated variables Y_{pq} are defined analogously. We will term these families of distributions the *quantile-restricted* distributions of X and Y . Finally, we will say that two distributions F_X and F_Y have the CMO property if either

$$E(X_{pq}) \geq E(Y_{pq}) \text{ for all } 0 \leq p < q \leq 1,$$

or

$$E(Y_{pq}) \geq E(X_{pq}) \text{ for all } 0 \leq p < q \leq 1.$$

In other words, if two distributions have the CMO property, then the ordering of the means for the unrestricted random variables characterizes the ordering of the means for all of their quantile-restricted distributions. If the CMO property holds and $E(Y) \geq E(X)$, then we will say that Y coherently dominates X in mean, or $Y \succ_m X$. Figures 2(a) and 2(b) show cases where CMO holds and is violated, respectively. In the following theorem, we show that coherent mean ordering is equivalent to ordinary first-order stochastic dominance.

An analogous property can be expressed for variance. We say that two distributions F_X and F_Y have *coherent variance ordering* (CVO) if either

$$\text{var}(X_{pq}) \geq \text{var}(Y_{pq}) \text{ for all } 0 \leq p < q \leq 1,$$

or

$$\text{var}(Y_{pq}) \geq \text{var}(X_{pq}) \text{ for all } 0 \leq p < q \leq 1.$$

If the CVO property holds and $\text{var}(Y) \geq \text{var}(X)$, then we will say that Y coherently dominates X in variance, or $Y \succ_v X$.

For instance, if the CVO property holds and RTs in a congruent condition are more variable than RTs in an incongruent

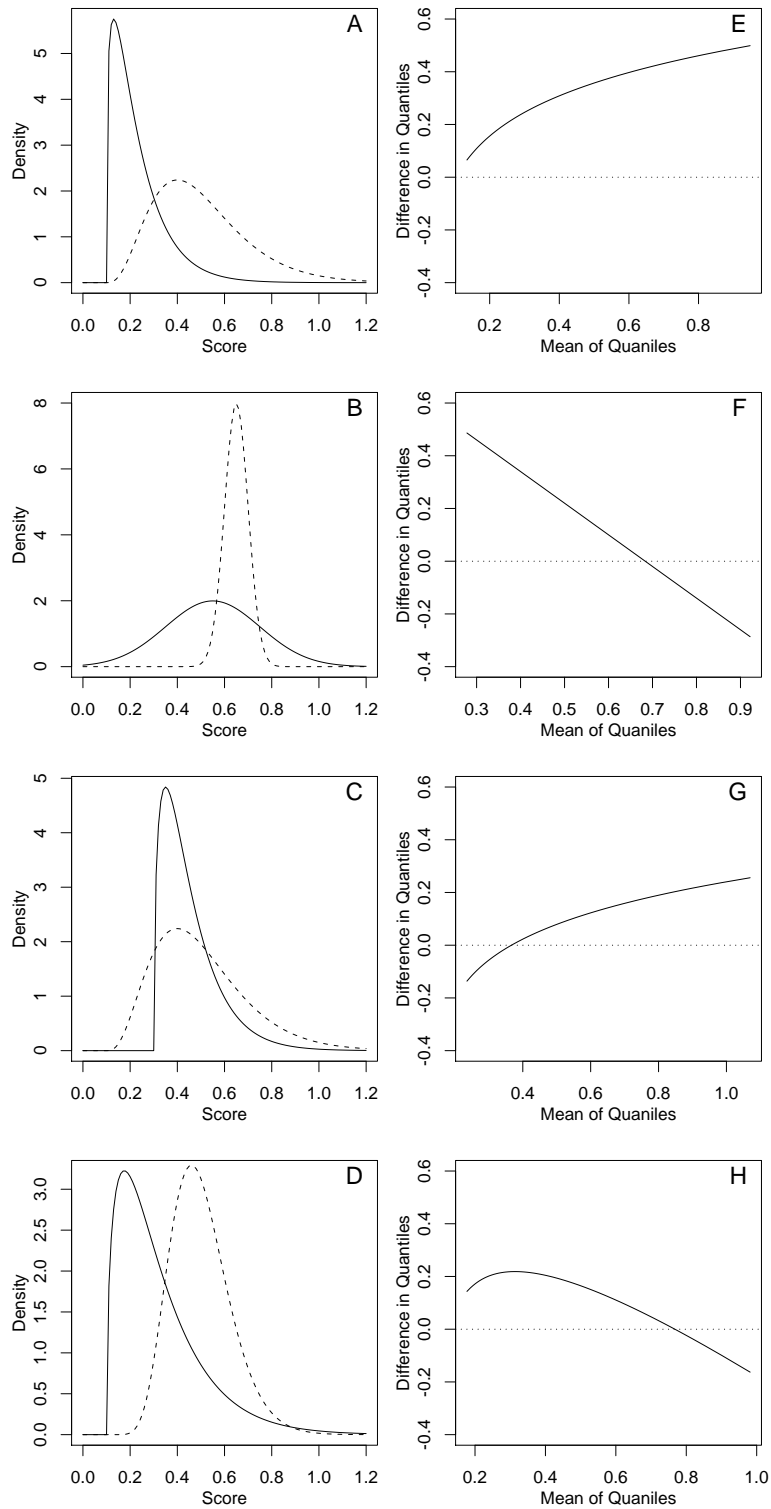


Figure 2. (a) and (b): Distributions for which coherent mean ordering holds and is violated, respectively. (c) and (d): Distributions for which the coherent variances ordering holds and is violated, respectively. (e)–(h): Delta plots for (a)–(d), respectively.

one, then there are no quantile intervals in which incongruent-condition RTs are more variable than congruent-condition RTs. Figure 2(c) and 2(d) show cases where the CVO property holds and is violated, respectively. Theorem 1 shows the relationship between the coherent ordering properties and the quantile functions of the two distributions.

2. CHARACTERIZING CMO AND CVO

The two properties, coherent mean ordering and coherent variance ordering, can be easily characterized in terms of the quantile functions of the two distributions. We assume that X and Y have continuous distributions with continuous probability density functions supported on possibly infinite intervals. These conditions imply that Q_X and Q_Y are continuously differentiable on $(0, 1)$.

Theorem 1. (a) A random variable Y coherently dominates X in mean if and only if

$$Q_Y(p) \geq Q_X(p) \quad \text{for all } 0 < p < 1, \quad (1)$$

that is, if and only if Y is stochastically larger than X .

(b) A random variable Y coherently dominates X in variance if and only if

$$Q'_Y(p) \geq Q'_X(p) \quad \text{for all } 0 < p < 1. \quad (2)$$

This theorem provides a relationship between delta plots and coherent orderings. Coherent mean ordering holds if and only if the delta plot is either entirely positive or entirely negative. Similarly, a consequence of (2) is that coherent variance ordering holds if and only if the delta plot is monotonic. In fact, if one plots quantile differences $Q_X(p) - Q_Y(p)$ against p directly, then coherent variance ordering clearly implies the plot is monotonic by (2). The delta plot is a graph of $Q_X - Q_Y$ against $m(p) = [Q_X(p) + Q_Y(p)]/2$. Since $d(Q_Y - Q_X)/dm = (Q'_Y - Q'_X)/(dm/dp)$ and $m(p)$ is strictly increasing, $Q'_Y - Q'_X$ and $d(Q_Y - Q_X)/dm$ have the same sign.

The equivalence between coherent mean ordering and stochastic dominance in part (a) of Theorem 1 is not surprising. Clearly $\lim_{q \downarrow p} E(X_{pq}) = Q(p)$, so CMO implies (1). Conversely, if strict inequality holds in (1) at some p_0 , it is not hard to show that $E(X_{pq}) < E(Y_{pq})$ for sufficiently small intervals $p_0 \in (p, q)$. This idea can be extended to prove the converse. An alternate proof is given in the Appendix.

3. RELATIONSHIP WITH STOCHASTIC DOMINANCE

The notion of stochastic dominance is familiar in the economics and finance literature (e.g., Levy 1992) as well as the statistics literature (e.g., Abadie 2002). The distribution of Y stochastically dominates (i.e., is stochastically larger than) that of X if $F_Y(t) \leq F_X(t)$ for all t . Under the conditions of Theorem 1, stochastic dominance is obviously equivalent to (1). Thus, Theorem 1 gives an intuitive equivalent condition for stochastic dominance.

Perhaps the simplest definition of ranking by conditional mean values would be to condition on fixed intervals, say $[a, b]$, rather than fixed quantiles. However, this ranking is not equivalent to FOSD. For example, suppose $Y \sim \text{uniform}[-1, 1]$ and X has pdf $p(x) = 3/4$ for $-1 \leq x \leq 0$, $p(x) = x/2$ for $0 < x \leq 1$, and $p(x) = 0$ elsewhere. It is easy to see that the distribution of Y is stochastically larger than that of X and $E(Y) > E(X)$. However, $E(Y | 0 \leq Y \leq 1) = 1/2 < E(X | 0 < X < 1) = 3/4$, so the conditional mean condition is violated.

Coherent variance dominance does not appear to be related to any other conventional notion of stochastic dominance. Under very special circumstances (such as normal distributions with equal means), ranking by unconditional variance is equivalent to second-order stochastic dominance (SOSD). [SOSD of a nonnegative random variable Y over another nonnegative random variable X holds if and only if

$$\int_0^z (1 - F_X(x)) dx \geq \int_0^z (1 - F_Y(y)) dy$$

for all $z > 0$ (e.g., Bawa 1975).] In contrast, for distributions obeying coherent variance ordering, ranking by unconditional variance is equivalent to coherent variance domination.

Once again, the use of quantile (i.e., probability) information appears to be crucial. For example, suppose $Y \sim \text{uniform}[-2, 2]$, so $Q_Y(p) = 2(2p - 1)$ and $Q'_Y(p) = 4$ for $0 < p < 1$. Next, let X have pdf $f(x) = 3(x^2 + 1)/8$, $-1 \leq x \leq 1$, and zero elsewhere. Because $f(x) \geq 3/8$ on the support of X , the quantile function satisfies $Q'_X(p) = 1/f(Q(p)) \leq 8/3$ for $0 < p < 1$, and Y coherently dominates X in variance. However, $\text{var}(Y | -1 < Y < 1) = 1/3$, which is less than $\text{var}(X | -1 < X < 1) = 2/5$. Thus the conditional variance condition on a fixed interval $[a, b]$ does not necessarily hold for CVO.

4. DISCUSSION

Consideration of coherent ordering and the associated delta plot has substantive, theoretical, and practical advantages. From a substantive viewpoint, violations of orderings imply complex relations such as the presence of multiple processes or mixtures. Satisfaction of orderings, in contrast, may be accounted for by simpler relations. From a theoretical viewpoint, Theorem 1 provides a novel relationship between the first and second moments and quantile functions. From a practical viewpoint, delta plots provide a quick and convenient method of assessing orderings.

APPENDIX

For the proof of the theorem, we need two lemmas.

Lemma 1. Suppose X is a random variable with continuous quantile function $Q(p)$. If $Q(p)$ is also differentiable a.e. with $Q'(p) > 0$ for almost all $p \in (0, 1)$, then

$$\begin{aligned} E(X_{pq}^k) &= \frac{1}{q-p} \int_{Q(p)}^{Q(q)} x^k f(x) dx \\ &= \frac{1}{q-p} \int_p^q Q(u)^k du, \quad 0 \leq p < q \leq 1, \end{aligned}$$

whenever the expectation exists.

Proof: Use the substitution $x = Q(u)$.

Lemma 2. *If the quantile function Q_X satisfies the conditions of Lemma 1, then*

$$\text{var}(X_{pq}) = \int_p^q \int_p^q Q'(s)Q'(t)h(s, t; p, q) ds dt,$$

$$0 \leq p < q \leq 1,$$

where

$$h(s, t; p, q) = \left(\frac{q - s \vee t}{q - p} \right) \left(\frac{s \wedge t - p}{q - p} \right).$$

Proof: First consider the case with $p > 0$, so $Q(p) > -\infty$. Since $\text{var}(X_{pq}) = \text{var}(X_{pq} - Q(p))$, without loss of generality, assume $Q(p) = 0$. Then

$$\int_p^q Q(u) du = \int_p^q \int_p^u Q'(s) ds du = \int_p^q (q - s)Q'(s) ds.$$

Similarly,

$$\begin{aligned} & \int_p^q Q(u)^2 du \\ &= \int_p^q \left\{ \int_p^u Q'(s) ds \int_p^u Q'(t) dt \right\} du \\ &= \int_p^q \int_p^q Q'(s)Q'(t) \left\{ \int_p^q 1_{[p,u]}(s)1_{[p,u]}(t) du \right\} ds dt \\ &= \int_p^q \int_p^q Q'(s)Q'(t)(q - s \vee t) ds dt. \end{aligned}$$

Using Lemma 1 with these two expressions,

$$\begin{aligned} \text{var}(X_{pq}) &= \frac{1}{q - p} \int_p^q \int_p^q Q'(s)Q'(t)(q - s \vee t) ds dt \\ &\quad - \left[\int_p^q \left(\frac{q - s}{q - p} \right) Q'(s) ds \right] \\ &\quad \times \left[\int_p^q \left(\frac{q - t}{q - p} \right) Q'(t) dt \right], \end{aligned}$$

which simplifies to the expression in the lemma.

For the general case, note that $\lim_{p \rightarrow 0} E(X_{pq}^k) = E(X_{0q}^k)$, $k = 1, 2$, whenever the latter variance exists.

Proof of the Theorem. For part (a), sufficiency follows directly from Lemma 1 with $k = 1$. On the other hand, using Lemma 1 again, $\lim_{q \downarrow p} E(X_{pq}) = Q(p)$, and necessity follows.

To prove part (b), note that $h(s, t; p, q) \geq 0$ for all s, t and for any $0 \leq p < q \leq 1$. Thus the condition of the theorem implies coherent variance domination. For the converse, first note that

$$\frac{1}{q - p} \int_p^q \int_p^q h(s, t; p, q) ds dt = \frac{q - p}{12}.$$

By assumption Q'_X and Q'_Y are both continuous. If Y coherently dominates X in variance,

$$\begin{aligned} Q'_Y(p)^2 &= \lim_{q \downarrow p} \frac{12}{q - p} \text{var}(Y_{pq}) \\ &\geq \lim_{q \downarrow p} \frac{12}{q - p} \text{Var}(X_{pq}) = Q'_X(p)^2. \end{aligned}$$

But both Q_X and Q_Y are nondecreasing functions, so $Q'_X(p)$ and $Q'_Y(p)$ are both nonnegative, proving the converse.

[Received January 2008. Revised April 2008.]

REFERENCES

- Abadie, A. (2002), "Bootstrap Tests for Distributional Treatment Effects in Instrumental Variable Models," *Journal of the American Statistical Association*, 97, 284–292.
- Bawa, V. S. (1975), "Optimal Rules for Ordering Uncertain Prospects," *Journal of Financial Economics*, 2, 95–121.
- De Jong, R., Liang, C. C., and Lauber, E. (1994), "Conditional and Unconditional Automaticity: A Dual-Process Model of Effects of Spatial Stimulus-Response Concordance," *Journal of Experimental Psychology: Human Perception and Performance*, 20, 731–750.
- Huber, D. E., Shiffrin, R. M., Quash, R., and Lyle, K. B. (2002), "Mechanisms of Source Confusion and Discounting in Short-Term Priming 1: Effects of Prime Duration and Prime Recognition," *Memory & Cognition*, 30, 745–757.
- Levy, H. (1992), "Stochastic Dominance and Expected Utility: Survey and Analysis," *Management Science*, 38, 555–593.
- Ridderinkhof, K. R. (2002), "Micro- and Macro-Adjustments of Task Set: Activation and Suppression in Conflict Tasks," *Psychological Research*, 66, 312–323.
- Stroop, J. R. (1935), "Studies of Interference in Serial Verbal Reactions," *Journal of Experimental Psychology*, 18, 643–662.
- Wilk, M. B., and Gnanadesikan, R. (1968), "Probability Plotting Methods for the Analysis of Data," *Biometrika*, 55, 1–17.