Bayes Factor Approaches for Testing Interval Null Hypotheses

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Abstract

Psychological theories are statements of constraint. The role of hypothesis testing in psychology is to test whether specific theoretical constraints hold in data. Bayesian statistics is well-suited the task of finding supporting evidence for constraint, because it allows for comparing evidence for two hypotheses against one another. One issue in hypothesis testing is that constraints may hold only approximately rather than exactly, and the reason for small deviations may be trivial or uninteresting. In the large-sample limit, these uninteresting, small deviations will lead to the rejection of a useful constraint. In this paper, we develop several Bayes factor one-sample tests for the assessment of approximate equality and ordinal constraints. In these tests, the null hypothesis covers a small interval of nonzero but negligible effect sizes around zero. These Bayes factors are alternatives to previously-developed Bayes factors, which do not allow for interval null hypotheses, and may especially prove useful to researchers who use statistical equivalence testing. To facilitate adoption of these Bayes factor tests, we provide easy-to-use software.

Bayes Factor Approaches for Testing Interval Null Hypotheses

The usefulness of psychological theories is determined by the extent to which they make constrained predictions about data. In many cases, these constraints are ordinal in nature: participants are predicted to perform better in one condition than another, or participants with a certain characteristic are predicted to perform better than those without this characteristic. In some cases, the predictions are stronger; they are about equalities. One example is the Fechner-Weber law, which describes the amount of intensity, ΔI, that must be added to a background of intensity I for successful discrimination. The law states that performance varies as $\Delta I/I$; the constraint is that when the intensity of an increment and background are both multiplied by the same constant, detection performance will be the same. Another example of a theoretically interesting equality constraint is the lack of effect of gender on certain tasks (Shibley Hyde, 2005, 2007) or a lack of interaction across specified manipulations (e.g., additive factors, Sternberg, 1969). Furthermore, the assessment of formal models – for example, structural equation models – is a matter of assessing equality constraints. The assessment of both equality and ordinal constraints is the primary use of hypothesis testing in the psychological sciences. In this paper, we develop Bayes factor methods for evaluating evidence for constraints in data. In the following development, we first focus on assessing evidence for equality constraints, and then generalize the approach to ordinal constraints.

Our development of Bayes factor tests for equality constraints is motivated by several common critiques of null-hypothesis significance tests (NHST). One critique of testing for equality constraints is that such constraints are not plausible; that is, small violations of equality constraints always exist (Cohen, 1994; Meehl, 1987). A second critique is that null hypothesis significance tests are asymmetric in that equality constraints may only be rejected and not accepted. These two critiques provide rationales for statistical equivalence tests, in which approximate equality constraints serve as the alternative hypothesis rather than the default, null hypothesis. The third critique is that the asymmetry in choosing default hypotheses, whether the null or alternative, leads to inference that overstates the evidence against the default hypothesis (Berger & Sellke, 1987; Rouder et al., 2009; Sellke, Bayarri, & Berger, 2001; Wagenmakers, 2007). This third critique serves as motivation for the use of Bayes factors. In this paper we develop Bayes factor tests that allow for equivalence regions.

Nil Hypotheses, Null Hypotheses, and Equivalence Regions

Equality constraints may be represented as point-null hypotheses. Without any loss of generality, any point hypothesis can be expressed as a *nil hypothesis*; that is, the statement that a parameter of interest is zero (Cohen, 1994)[1]. Nil hypotheses are statements such as: one variable has no effect on an another; there is no difference between two variables; there is no association between two variables; or, the parameter has a specific value. In order to mitigate confusion, we will use the following terminology throughout: the *nil hypothesis* refers to the point hypothesis that the parameter is identically 0. The *null hypothesis* is more general: it may be restricted to a nil hypothesis or may allow for values which deviate slightly from the nil. The null hypothesis therefore may capture nonzero effects that are too small to be of interest. A third type of hypothesis, the *default hypothesis*, refers to a hypothesis assumed true unless sufficient evidence is found against it. The nil hypothesis is the default hypothesis in most null hypothesis significance tests, but not in statistical equivalence testing.

Cohen (1994), among others (Berger & Delampady, 1987; Berkson, 1938; Meehl, 1978), offers the viewpoint that nil hypotheses never hold to an arbitrary level of precision; that is, *a priori* nil hypotheses are false. For instance, the Fechner-Weber law, discussed above cannot hold to arbitrary precision because at some level, the process becomes discretized (if for no other reason than light itself is quantal in nature). As another example, a researcher might be interested in whether a particular genetic mutation correlates with educational outcomes. Even if the gene codes for something unrelated – say, hair growth rate – it seems unlikely that in the population, people with the mutation have *exactly* the same educational outcomes as those without. The main point in these examples is that the nil will fail for trivial or uninteresting reasons. Meehl (1978) refers to the trivial relationships that variables have to others as the "crud factor" in psychological research.

In this paper, we take as a default position that the nil never holds to arbitrary precision. The fact that nil hypotheses may break down, however, is not an argument against their usefulness in many areas of psychological research. Often these nil hypotheses are good approximations and are exceedingly useful for guiding theory development. For instance, even though the Fechner-Weber Law breaks down for very low intensities, it may nonetheless provide appropriate constraint on the mechanisms of perceptual decision making. Likewise, even though the equality of men and women in the performance of a certain task may break down in some limit, it is still useful in suggesting that the mental processing underlying the task may not vary across gender. Hence, the argument that the nil hypothesis is never exactly correct is not an argument against proposing constraints. The fact that the nil hypothesis is never true is, however, an important consideration when performing statistical tests. A key goal in our development are methods that do not reject null hypotheses if the failure is due to trivially small effects.

One solution is to abandon point hypotheses altogether. Hodges and Lehmann (1954) suggested establishing a region of parameter values around the nil hypothesis that are not "materially significant" – that is, although the true parameter value may indeed be slightly different from the nil hypothesis value, the nil hypothesis would still be preferred for parsimony's sake. This same logic is adopted by the statistical equivalence test (SET) paradigm (Rogers, Howard, & Vessey, 1993; Wellek, 2003). In SET, an "equivalence" region is defined on a parameter of interest. This equivalence region contains all parameter values which may be considered too small to be meaningfully different from 0. The researcher specifies this equivalence region before analysis and its size is informed by substantive considerations in the domain at hand.

One way to formulate a statistical equivalence test is with confidence intervals, as shown in Figure 1. The equivalence region in the figure is defined from -0.2 to 0.2. All parameter values within these bounds are considered "equivalent" to 0. The default hypothesis is nonequivalence: that is, the parameter is outside the equivalence region. Because the nonequivalence hypotheses is default, we reject nonequivalence only with sufficient evidence and retain nonequivalence otherwise. Evidence in this formulation is the relationship of confidence intervals to the equivalence bounds. If the $100(1-2\alpha)\%$ confidence interval lies within the equivalence region, then nonequivalence is rejected. Shown in the figure are hypothetical CIs for four samples. The confidence intervals for Samples A, B, and D do not provide sufficient evidence to reject nonequivalence. The confidence interval for Sample C, however, only contains values in the equivalence region. For Sample C, the nonequivalence null hypothesis may be rejected in favor of equivalence.

Relative Evidence

The methods discussed above are all based on the idea of controlling the rates of certain kinds of errors. This is the traditional NHST approach; Type I error rate is held at a rate of α, or CIs are built which fail to cover the true value at a rate of α. However, it is often useful in scientific discourse to have methods which measure the evidence for positions rather than make a binary choice between them. In both Bayesian and freqentist statistics, the likelihood ratio is used as a measure of relative evidence between two hypotheses (Royall, 1997). Measuring relative evidence is fairly straightforward if both hypotheses are points. For instance, suppose wish to compare the evidence for the hypothesis that $\mu = \mu_a$ versus the hypothesis that $\mu = \mu_b$ in a one-sample design. The two hypotheses may be written as two models:

$$\begin{aligned} H_a: \quad & y_i \overset{iid}{\sim} \text{Normal}(\mu_a, \sigma^2), \\ H_b \quad & y_i \overset{iid}{\sim} \text{Normal}(\mu_b, \sigma^2), \end{aligned}$$

where μ_a, μ_b, and σ^2 are prespecified values. The likelihood ratio quantifying the evidence for hypothesis H_a relative to hypothesis H_b, which we abbreviate B_{ab}, is

$$B_{ab} = \frac{Pr(\boldsymbol{Y} \mid H_a)}{Pr(\boldsymbol{Y} \mid H_b)}, \tag{1}$$

where $\boldsymbol{Y} = (y_1, y_2, \ldots, y_N)$ is a vector of observations. Because the data are assumed to be normally distributed and conditionally independent, we use the product of normal distribution functions[2] $\phi(y \mid \mu, \sigma^2)$ to form the likelihood ratio:

$$B_{ab} = \frac{\prod_{i=1}^{N} \phi(y_i \mid \mu_a, \sigma^2)}{\prod_{i=1}^{N} \phi(y_i \mid \mu_b, \sigma^2)}.$$

The notion that relative evidence is provided by the likelihood ratio is used in both frequentist and Bayesian paradigms.

Because most hypothesis tests in psychological research are built on the premise of controlling Type I rates, rather than on the basis of evidence, it is interesting to ask how much relative evidence is there in rejections of hypotheses in NHST testing (Lindley, 1957, Sellke, Bayarri, & Berger, 2001). For example, consider the case of a researcher who observes a one-sided t value of 1.75 with 30 observations. We take as H_b the nil that $\mu = 0$ and as H_a an alternative that $\mu = .2\sigma$ (that is, the standardized effect size is .2), where σ, the standard deviation, is an unknown. In this case, $Pr(\boldsymbol{Y}|H_b)$ is the density at 1.75 under the central t distribution. The $Pr(\boldsymbol{Y}|H_a)$ is the density at 1.75 under a noncentral t with noncentrality parameter $.2\sqrt{30}$. These areas are shown in Figure 2, and the relative evidence in this case, the ratio, is $B_{ab} = 3.55$. What strikes us is that the relative evidence is so small in favor of the alternative given that null is rejected.

The above evidence ratio is limited to the comparison of a nil and a specific alternative in which the effect size is .2. Instead of asking about a single prespecified alternative, we could ask if there is some other alternative in which there is sizable evidence for the alternative. In fact, one can compute the relative evidence for all other effect sizes against the nil, but for the observed p value and N no alternative achieves a relative evidence of more than about 4.4 against the nil. Even though researchers may reject the nil with significance testing, there is no single alternative for which the data are dramatically more probable.

It is also useful to assess the relative evidence in SET, and by extension, inference by CI. Although SET has no p values to compare, we can examine relative probabilities of rejecting nonequivalence. Suppose we collect a sample of 70 observations, and on the basis of these data, reject nonequivalence with an equivalence region of [-0.2,0.2]. If the true mean were just outside the equivalence region, we would only reject nonequivalence about .56% of the time. However, rejecting nonequivalence is not that much more common under the best possible equivalence scenario, $\mu = 0$. If $\mu = 0$, then we will correctly reject nonequivalence only 2.2%. This, the evidence ratio provided by our rejection of nonequivalence is only $2.2/.56 \approx 4$. Our rejection of the nonequivalence was based on relatively light evidence.

Bayes Factors

In Bayesian inference, the relative evidence measure B is known as the Bayes factor. We previously developed the Bayes factor for simple point hypotheses. In most applications, however, alternative hypotheses are composites in which parameters are assumed to take on a range of values. For example, suppose under the alternative H_a the value of μ is allowed to range across all real numbers. In Bayesian statistics, probability is interpreted as uncertainty; to compute the probability of data under a hypothesis, we average over the uncertainty in the parameters. The probability of data under the hypothesis is thus:

$$P(Y|H_a) = \int_{\mu} P(\mathbf{Y}|\mu)\mathbf{f}(\mu)\mathbf{d}\mu. \tag{2}$$

The term $P(\mathbf{Y}|\mu)$ is the conditional probability density for the data given the parameter μ (the likelihood), and the term $f(\mu)$ is the prior probability density for μ. A more general formulation, for when a vector of parameters is free to vary under a hypothesis H, is

$$P(Y|H) = \int_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} P(\boldsymbol{Y}|H,\boldsymbol{\theta})f(\boldsymbol{\theta}|H)d\boldsymbol{\theta}, \tag{3}$$

where $\boldsymbol{\theta}$ is a vector of parameters, and $\boldsymbol{\Theta}$ is the space across which the parameters vary. The function f is the *prior density* of $\boldsymbol{\theta}$ and describes the researchers belief or uncertainty about the parameters before observing the data. In Bayesian analysis, the specification of f is critical to defining the hypotheses, and is the point where subjective probability enters Bayesian inference. We realize that frequentists may object to the conceptualization of probability as capturing a degree or measure of belief. The argument for subjective probability is made most elegantly in the psychological literature by Edwards et al. (1963). We note here that subjective probability stands on firm axiomatic foundations, and leads to ideal rules about updating beliefs in light of data (Cox, 1946; De Finetti, 1992; Gelman, Carlin, Stern, & Rubin, 2004; Jaynes, 1986).

The marginal probability in Eq. (3) may be viewed as a weighted average across all parameter values, with the weights for each parameter value is given by a prior distribution. Priors that place greater weight on parameter values not concordant with the data have lower marginal probability. This weighted averaging properly penalizes flexibility in models in which a large set of parameters have some *a priori* weight (Myung, 2000). Flexible models may fit the data better (i.e. be preferred by the likelihood ratio) than more constrained ones, but they will also include prior weight over unlikely parameter values. Averaging over the prior ensures that flexible models are properly penalized for their flexibility.

This averaging across all parameter values also reveals a key difference between Bayesian methods and frequentist methods: the interpretation of probability as uncertainty licenses averaging. Parameters are treated as random quantities which have distributions that may be marginalized. In frequentist methods, however, this averaging is not licensed; other approaches, such as computing maximum likelihood values, are used.

In Bayesian inference, it is also possible to compute the odds of competing two hypotheses against one another, given the observed data; e.g., $P(H_a|\boldsymbol{Y})/P(H_b|\boldsymbol{Y})$. This quantity, the *posterior odds* is related to the Bayes factor:

$$\frac{P(H_a|\boldsymbol{Y})}{P(H_b|\boldsymbol{Y})} = B_{ab} \times \frac{P(H_a)}{P(H_b)}, \tag{4}$$

where $\frac{P(H_a)}{P(H_b)}$ is the prior odds and describes the relative degree of belief in the hypotheses before the data are observed. The Bayes factor describes how beliefs are to be updated. Jeffreys (1961) and Kass (1993) note that the Bayes factor is ideal for reporting evidence because it describes how researchers should change their beliefs regardless of what those initial beliefs are. Prior odds provide readers and analysts a mechanism to add context as to how evidence is to be interpreted. We wish to emphasize, however, that the Bayes factor may be interpreted as the relative evidence contributed by the data, without stipulating prior odds.

Bayes factor for nil hypotheses

In constructing Bayes factors, different hypotheses may be implemented as different choices of priors over parameters. For example, the nil hypothesis may be implemented by specifying a point prior over the parameter of interest. More interesting are the priors under the alternative. It is common when using Bayesian techniques for parameter estimation to assume diffuse or "noninformative" priors (Gelman, Carlin, Stern, & Rubin, 2004) for these alternatives. These diffuse priors are used to minimize the influence of the prior distribution on the posterior distribution. Noninformative priors are often improper, meaning that they do not integrate to a finite number. A common improper, noninformative prior for the mean of a normal distribution, for instance, is a constant value over all real numbers. This constant prior corresponds to the assumption of absolute ignorance; that is, that no value for the mean is *a priori* more likely than any other. As long as the posterior distribution is proper, the impropriety of the prior poses no problem for Bayesian parameter estimation.

For hypothesis testing, as opposed to estimation, the choice of a diffuse prior for the alternative hypothesis is problematic. The reason is that the Bayes factor is the ratio of the expectation of the likelihoods, taken over their respective priors. As the prior becomes more and more diffuse, unlikely values dominate, making the expectation of the likelihood approach 0. If the null hypothesis is a point, and the alternative prior is noninformative, the point will be preferred, regardless of the data. Ironically, using "noninformative" priors in Bayes factors leads to the predetermined result of always choosing the null hypothesis (Lindley, 1957).

Jeffreys (1961) suggests placing priors on the standardized effect size, $\delta = \mu/\sigma$. In this case, $y_i \sim \text{Normal}(\sigma\delta, \sigma^2)$. As Iverson et al. (2009) note, effect size has a natural scale and proper priors for it are appropriate. The prior on the point nil is simply that the effect size is exactly zero. One reasonable prior on the alternative might be that effect sizes are distributed as a standard normal. This prior structure is inspired by the knowledge that true effect sizes are typically not large, and that the analyst is agnostic to the direction of any possible effect.

Although the standard normal is a reasoned choice, it is a thin-tailed distribution; that is, large effect sizes, such as 5.0, are effectively excluded from consideration. A less-informative prior would not effectively preclude the possibility of large effect sizes. To accomplish this goal, we use a distribution with fatter tails: the t distribution, as a prior on effect size. As the degrees of freedom of the t distribution decrease, the tails of the t distribution become fatter; with only 1 degree of freedom, the tails of the t distribution are fat enough that the distribution has no expected value or higher-order moments. The $t(1)$ distribution is also known as the standard Cauchy distribution. The standard Cauchy-distributed prior on δ quantifies an assumption that excessively large true δ parameters are much less plausible than small effect sizes, but are still possible. We express the prior structure as follows:

$$\begin{aligned} H_0 &: \delta = 0 \\ H_1 &: \delta \sim \text{Cauchy}(r = 1), \end{aligned}$$

where r is a scaling parameter, and represents one-half the interquartile range of the Cauchy. A Cauchy distribution with $r = 1$ is a standard Cauchy distribution.

A prior is also needed for σ^2. Because σ^2 is a parameter in both models, a noninformative prior on it is both practical and desirable:

$$p(\sigma^2) \propto \frac{1}{\sigma^2},$$

where $p(\sigma^2)$ represents an improper prior density on σ^2. The advantage of the $1/\sigma^2$ prior is that it is the Jeffreys prior. Jeffreys priors have the desirable property that they impart the same information, even if under transformations of parameter (Jeffreys, 1946).

Bayarri and Garcia-Donato (2007) called the combination of these priors on δ and σ^2 the Jeffreys-Zellner-Siow (JZS) prior to honor the contributions of Jeffreys (1961) and of Zellner and Siow (1980), who extended the prior to the class of linear models. The JZS model is depicted graphically in Figure 3A. The vertical line denotes the nil hypothesis at $\delta = 0$; the density is that of a Cauchy and denotes the prior on effect size size for the alternative hypothesis. Rouder et al. (2009) call the associated Bayes factor the JZS

Bayes factor. They provide expressions for the JZS Bayes factor for one- and two-sample tests as well as a web applet for computation (http://pcl.missouri.edu/bayesfactor).

Consider the following real-world example of the differences between JZS Bayes factor and NHST analysis. It is well-known that emotionally-evocative words are better recognized in a memory experiment than emotionally neutral words (Murphy & Isaacowitz, 2008). Although this increase in performance would seemingly indicate better memory for these stimuli, this pattern may instead reflect response biases (Dougal & Rotello, 2007; Thapar & Rouder, 2009). Thus, memory may be the same for both emotional valence levels, but participants are simply biased to state that emotionally-evocative words were studied more often than neutral words when guessing or when responding with limited information. Thapar and Rouder, for example, fit a variant of Luce's Similarity Choice Rule (Luce, 1963) and found emotional valence effects on response bias parameters but not in memory sensitivity parameters.

Grider and Malmberg (2008, Experiment 3) provide the following insightful test of the equality of memory across emotional valence levels. Participants first study both evocative and neutral words. At test, participants see a new word (the lure) as well as a studied word (the target), and have to judge which was studied. In Grider and Malmberg's task, lures and targets always had the same emotional valence so that each would be chosen equally likely in the absence of mnemonic information. With response bias controlled, it is reasonable to conclude that an approximate equality of performance supports the invariance of memory across emotional valence, and differences in performance support differences in memory across emotional valence. They find mean recognition accuracy for positive words (80%) was somewhat higher than mean recognition accuracy for neutral words (76%). They interpreted these data with the aid of a t test, finding a statistically significant effect of word valance ($t(79)$=2.24, $p = 0.014$, $\hat{\delta} = 0.25$). In fact, Grider & Malmberg used this result as part of their argument that memory varies across emotional valence levels.

The JZS Bayes factor for these data, on the other hand, is 1.02, indicating no preference for either a valence equality or a valence effect. The no-preference result here comes about because the relative evidence is equivocal. The difference in means, 0.04, is quite small (effect size of .25), and is unlikely under both the null or under an alternative that effects have reasonable variance away from zero (the JZS alternative). Accordingly, the rejection of the nil based on the *p* value is unwarranted because it does not account for how likely the data are under reasonable alternatives. This readiness to reject the nil hypothesis based on slight evidence is especially prominent for NHST of small effects with large sample sizes (Grider and Malmberg's sample size is, in fact, atypically large for a repeated-measures design in cognitive psychology. More typical sample sizes are approximately 25 participants).

One advantage the JZS Bayes factor has is consistency. In the large sample limit, the JZS ratio will appropriately converge to infinity or 0 depending on whether the nil or alternative is true. Consistency is not a property of conventional tests, because if the nil is true in NHST, it can never be accepted, regardless of how much data is collected[3]. Although consistency is an important property, the consistency of the JZS Bayes factor has a critical drawback. The JZS null hypothesis is the nil hypothesis on the parameter of interest. Hence, the relative evidence against the null hypothesis will grow without bound as sample size increases unless the nil hypothesis is *exactly* true. If we adopt Cohen's proposition that nil hypotheses are always false, albeit sometimes for uninteresting or trivial reasons, then the JZS Bayes factor will always yield support for the alternative, with large enough sample size. In this regard, the JZS Bayes factor shares an unfortunate property of NHST: it provides no means of assessing whether rejections of the nil are due to trivial or unimportant effect sizes or are due to more substantial effect sizes.

Bayes factor solutions to the nil hypothesis problem

In this section, we present three modifications of the nil hypothesis using the JZS prior. These modifications are motivated by the concern that the nil hypothesis is never true to arbitrary precision and is, therefore, inappropriate.

Overlapping hypotheses

Null hypotheses need not be restricted to nil hypotheses. Rouder et al. suggest a Cauchy-distributed null hypothesis, but one with a much smaller scale than the alternative (see Figure 3B). This null distribution is a distribution of effect sizes from trivial or uninteresting causes. The resulting models are:

$$\begin{aligned} y_i &\sim \text{Normal}(\sigma\delta, \sigma^2) && (5) \\ \delta &\sim \text{Cauchy}(r_i), && (6) \\ p(\sigma^2) &\propto \frac{1}{\sigma^2}, && (7) \end{aligned}$$

where i indexes hypotheses. Researchers must specify r_i for both the null and alternative distributions prior to analysis. Specifications with r_0 much less than r_1 will be the most useful as these capture the intuition that the spread of the negligible effect sizes is much smaller than those under the alternative. The JZS priors result as $r_0 \to 0$ and $r_1 = 1$. Because for any $r > 0$, the prior distribution of parameter under the null and alternative share common support, we call these priors the *overlapping-hypotheses priors* and the resulting Bayes factor as the overlapping-hypotheses (OH) Bayes factor. Rouder et al (2009) briefly mention this prior, but provide no development or analysis.

The computation of the OH Bayes factor is straightforward. Bayes factors are transitive in the following sense: let H_1, H_2, and H_3 be three hypotheses. The Bayes factor for H_1 relative to H_3, B_{13} is

$$B_{13} = B_{12}B_{23}.$$

Rouder et al. provide expressions for the Bayes factor of the the nil vs. a Cauchy with scale r, which is denoted here as $B_{01}(r)$. The OH Bayes factor for the null vs. alternative is computed by transitivity: $B_{01} = B_{01}(r_0)/B_{01}(r_1)$.

The previous example from Grider and Malmberg (2008) provides an opportunity to compare the JZS and OH Bayes factors in a real-world data set. The JZS Bayes factor was 1.02, which is nearly equal evidence for both hypotheses. To implement the OH Bayes factor we adopt the JZS setting for the alternative of $r_1 = 1$. For illustration purposes, we set the null to have a 1/10 the scale of the alternative, $r_0 = 0.1$. Under this alternative hypothesis, 50% of the effect sizes are between -0.1 and +0.1. With these prior settings, the resulting OH Bayes factor is $B_{01} = 2.25$. The data are over twice as likely to have come from the null hypothesis of negligible effects as from the alternative hypothesis of substantive ones. In this case, OH Bayes factors are more weighted toward the null than JZS Bayes factors. The reason for this is that the observed effect size $\hat{\delta} = 0.25$ is more consistent with the narrow $r = 0.1$ Cauchy distributed null than the point null.

Although the OH priors appear reasonable to account for negligible effects under the null, there is a subtle but important problem in interpretation. In the JZS priors, there was an unambiguous correspondence between true effect sizes and hypotheses. A true effect size of exactly zero corresponded to only the nil; nonzero true effect sizes corresponded to only the alternative. In the OH priors, this correspondence does not hold. Because both hypotheses share a common support, any effect size may come from either hypothesis. Even if a researcher knows the true effect size with absolute precision, it is not possible to decide with certainty between the null and alternative hypotheses. Consequently, the Bayes factor converges in the large sample limit to a finite nonzero value. The dotted line in Figure 4 shows this behavior for the case that true $\delta = 1$, $r_0 = 0.1$ and $r_1 = 1$. The lines represent the Bayes factors for the median t value that will be observed for an effect size $\delta = 1$ as sample size grows. The OH Bayes factor approaches the ratio of the prior densities at $\delta = 1$. In contrast, the JZS Bayes factor converges to $BF_{01} = 0$, indicating that increasing the sample size will eventually lead to one hypothesis, or the other, with certainty.

This undesirable limiting behavior of the OH Bayes factor is a direct consequence of the fact that the hypotheses were constructed as overlapping hypotheses. Any given effect size corresponds, to some extent, with both hypotheses. The ambiguity of true effect sizes relative to the hypotheses is undesirable. To mitigate this problem, we develop hypotheses that have exclusive support.

Non-overlapping hypotheses

The OH Bayes factor has properties that make it unsuitable for inference. The same true effect size could be considered null, or not null, depending on whether it came from the null or alternative distribution. This interpretability problem, along with the consequence that the Bayes factor does not converge to either 0 or ∞, motivates the development of a Bayes factor in which the null and alternative are mutually exclusive ranges of values. We specify a set of priors that are non-overlapping and derive the corresponding *non-overlapping* (NOH) Bayes factor as follows. To begin, consider the model:

$$y_i \sim \text{Normal}(\sigma\delta, \sigma^2) \tag{8}$$

$$\delta \sim t(\nu_0) \tag{9}$$

$$p(\sigma^2) \propto \frac{1}{\sigma^2}. \tag{10}$$

In this case, the model has been slightly modified so that the distribution on δ is t rather than scaled Cauchy. The researcher must choose a value for ν_0, the degrees of the t distribution. Setting $\nu_0 = 1$ yields the JZS Cauchy prior; setting $\nu_0 = \infty$ yields a standard normal prior (the unit-information prior of Rouder et al., 2009). This generalization allows the model to subsume the two models suggested by Rouder et al.

The null hypothesis H_2 for the nonoverlapping (NOH) Bayes Factor is that the effect size δ is within some range $(-c, c)$ of 0. The alternative H_3 is that the effect size is not within this range.

$$\begin{aligned} H_2 &: \delta \sim t(\nu_0), \delta \in (-c, c) \\ H_3 &: \delta \sim t(\nu_0), \delta \notin (-c, c) \end{aligned}$$

The NOH Bayes Factor B_{23} is

$$\mathrm{B}_{23} = \frac{\int_{\delta \in \Delta_2} \int_{\lambda^2} (\lambda^2)^{-\frac{N}{2}-1} \exp\left\{ -\frac{N-1}{2\lambda^2} - \frac{1}{2\lambda^2}\left(t - \delta\sqrt{N\lambda^2}\right)^2 \right\} \left(1 + \frac{(\delta/r)^2}{\nu_0}\right)^{-\frac{\nu_0+1}{2}} d\lambda^2 d\delta}{\int_{\delta \in \Delta_3} \int_{\lambda^2} (\lambda^2)^{-\frac{N}{2}-1} \exp\left\{ -\frac{N-1}{2\lambda^2} - \frac{1}{2\lambda^2}\left(t - \delta\sqrt{N\lambda^2}\right)^2 \right\} \left(1 + \frac{(\delta/r)^2}{\nu_0}\right)^{-\frac{\nu_0+1}{2}} d\lambda^2 d\delta} \times \frac{1 - \pi_2}{\pi_2}$$

where Δ_2 is the null region, and Δ_3 is the complement of Δ_2. The derivation of this formula is shown in the appendix. To compute the Bayes factor B_{23} requires 3 integrals: first, the integration with respect to λ^2, then the integration with respect to δ for each hypothesis. A closed form expression for the Bayes factor is not available, but the integrals may be performed numerically. We discuss methods of computing this integral in the appendix, and provide a convenient web applet for computing Bayes factors at http://pcl.missouri.edu/bayesfactor.

Like the JZS Bayes Factor B_{01}, the NOH Bayes Factor B_{23} is a function of only the observed t and the sample size N. The researcher must supply the bounds of the null hypotheses region $(-c, c)$; this will be discussed later. Following Cohen's suggestion (Cohen, 1988) that 0.2 is a "small" effect size, our examples all use a null region of (-0.2, 0.2).

Figure 5 shows the NOH Bayes factor for the median observed t value for several sample sizes and true effect sizes. The Bayes factor converges to the correct hypothesis for all hypotheses except when the true effect sizes is exactly on a boundary (-0.2 and 0.2). In this case, the Bayes factor converges a finite constant which depends on the prior distributions, because the data cannot differentiate between the two hypotheses. The figure includes the NOH Bayes factor for the two extreme values of ν_0, 1 and ∞, corresponding respectively to a standard Cauchy prior and standard normal prior on δ. The Cauchy prior marginally favors the null hypothesis, due to the heavier weighting of larger effect sizes, but both priors lead to substantially the same result.

As a comparison to the JZS and OH Bayes factor, we return to the data of Grider and Malmburg (2008). Recall that the observed $t(79)$ was 2.24, and the observed effect size was $\hat{\delta} = 0.25$. Because the effect size is near the null effect range, we expect the NOH Bayes factor to be more favorable to the null hypothesis than the other two Bayes factors. Indeed, $B_{23} = 3.63$ indicating that the data show a reasonable amount of evidence for the hypothesis that the true effect size is within $(-0.2, 0.2)$. Whether Grider and Malmburg would consider all null hypotheses in this interval to be negligible is not known, but the Bayes factor nevertheless indicates that whatever effect exists is likely to be small.

A hybrid model

Previously, we have argued that (a) the *nil* point-null hypothesis is unlikely, and perhaps impossible, and (b) there is a range of effect sizes around the nil hypothesis that, from the researcher's perspective, should be treated as null. If we accept (a), then (b) naturally follows. However, we may accept (b) but deny (a): the two claims need not be accepted together. This fact motivates the following generalization of the NOH null. Consider the mixture model for the null shown in Figure 3D. Like the JZS point-null model, there is point mass at $\delta = 0$. However, like the NOH null model, small effect sizes around the nil hypothesis are also considered null. We call this model the *hybrid null model*. Let π_0 be the mixing probability; $\pi_0 = 1$ corresponds to the case where the null mixture consists solely of the JZS null and $\pi_0 = 0$ corresponds to the case where the null mixture consists solely of the NOH null.

We call the associated Bayes factor the *hybrid Bayes factor* and denote it $B_{(02)3}$ to indicate that the null is the mixture of H_0 (the nil) and H_2 ($\delta \in (-c, c)$) while the alternative is H_3 ($\delta \notin (-c, c)$):

$$B_{(02)3} = \frac{P(Y|H_0 \text{ or } H_2)}{P(Y|H_3)}.$$

Computation of $B_{(02)3}$ is relatively straightforward. Because H_0 and H_2 are mutually exclusive, the hybrid Bayes factor may be written

$$B_{(02)3} = \frac{\pi_0 P(Y|H_0) + (1 - \pi_0) P(Y|H_2)}{P(Y|H_3)}$$

where

$$\pi_0 = \frac{P(H_0)}{P(H_0) + P(H_2)}$$

The constant π_0 acts as a weight that must be determined before the analysis. It represents the analyst's prior beliefs about the proportion of null hypotheses that are nil. Because the hypotheses H_2 and H_3 together are the same as the alternative H_1 of the JZS Bayes factor,

$$P(Y|H_1) = \pi_2 P(Y|H_2) + (1 - \pi_2) P(Y|H_3)$$

Using the above facts, algebraic rearrangement yields

$$B_{(02)3} \quad = \quad \pi_0 B_{01}(\pi_2 B_{23} - \pi_2 + 1) + B_{23}(1 - \pi_0), \tag{11}$$

where

$$\begin{aligned} B_{01} &= \frac{P(Y|H_0)}{P(Y|H_1)} \\ B_{23} &= \frac{P(Y|H_2)}{P(Y|H_3)} \end{aligned}$$

Thus, the hybrid Bayes factor $\mathrm{B}_{(02)3}$ is only a function of the point-null Bayes factor, the corresponding NOH Bayes factor, and the mixing probability π_0.

The hybrid Bayes factor model may appear unusual at first glance. However, it has some interesting properties that make it an attractive model for inference. First, it is a generalization of both the JZS Bayes factor and the NOH Bayes factor. With $c \to 0$, the JZS Bayes factor is obtained. With $\pi_0 \to 0$, the NOH Bayes factor is obtained. Parameters of the hybrid model may be manipulated to obtain other interesting models as well, including tests of ordinal hypotheses to be discussed later.

Another interesting feature of the hybrid model can be seen by setting $\pi_0 = 1$. This model is shown in Figure 7A. The model represents the situation where the researcher wants to ignore small effect sizes in testing a nil hypothesis. This is a direct test of "reasonable" effect sizes against the nil. Figure 7B shows the corresponding hybrid Bayes factor, which we call the *notched Bayes factor*. Johnson and Rossell (2010) proposed similar priors on the alternative hypothesis for computing Bayes factors, arguing that these priors balance the rates of evidence accumulation for the nil and the alternative. This can be easily seen by comparing the rate of evidence accumulation for the nil hypothesis of the JZS Bayes factor to the rate of accumulation of the hybrid Bayes factor in Figure 7B. The notched Bayes factor follows the JZS Bayes factor fairly closely for moderate to large effect sizes, but accumulates evidence for the nil more quickly for small effect sizes.

For researchers who want to include negligible effect sizes with the null hypothesis, this is possible by making $\pi_0 < 1$. Figure 6 shows the hybrid Bayes factor for $\pi_0 = 0.5$ and $c = 0.2$. Like the NOH Bayes factor, the hybrid Bayes factor has desirable convergence property for all true effect sizes except those exactly on the boundary values.

Using the hybrid Bayes factor in the analysis of Grider and Malberg's data, we may take two approaches: $\pi_0 = 1$, for the notched model in Figure 7A, and $\pi_0 < 1$. The notched Bayes factor yields $BF_{(02)3} = 1.36$. This represents more evidence for the nil than the JZS Bayes factor, because negligible effect sizes don't count towards the alternative. On the other hand, it represents less evidence for the null than the NOH Bayes factor, because negligible effect sizes don't count towards the null.

We can also compute the Bayes factor with both a point-null and an interval null. With $\pi_0 = 0.5$, we obtain a hybrid Bayes factor $BF_{(02)3} = 2.50$. Because the hybrid Bayes factor includes effect sizes around the nil as evidence for the null hypothesis, this hybrid Bayes factor indicates more evidence for the null hypothesis than the JZS Bayes factor. Because it contains a point-null, it is indicates less evidence for the null than the NOH Bayes factor.

Extensions to Ordinal Constraint

The preceding development focused on testing theories that predict equality constraints. Some theories, however, predict only an ordering of covariate effects. In general, Bayes factor is well-suited for assessing these ordinal constraints, and the Bayes factor priors discussed previously may be extended in a straightforward manner.

Consider a researcher whose theory predicts a positive effect. If we assume that small negligible effects may be found, even when the theory is false (Meehl, 1978), then it is appropriate to set a lower limit on positive effects. For example, a researcher may consider all effects above 0.2 as concordant with the hypothesis and all those below .2 as inconcordant. Using standard NHST terminology, we call $\delta < .2$ the null hypothesis (H_0) and the researcher's hypothesis, $\delta \geq .2$, the alternative (H_1), although the terms are arbitrary in this case. Note that in this setup, negative effects are considered plausible under the null. The computation of this one-sided Bayes factor is a straightforward extension of the previous development. The one-sided Bayes factor is equivalent to the NOH Bayes factor with a null region of $(-\infty, c)$. All other details remain the same. We provide an applet to compute this one-sided Bayes factor at http://pcl.missouri.edu/bayesfactor, and software at http://drsmorey.org/research/rdmorey.

The value of this one-sided Bayes factor for the Grider and Malberg data is 0.41. When the Bayes factor is less than 1.0, it is often more convenient to report the reciprocal and the favored hypothesis, which in this case is about 2.46 in favor of the the alternative. The Bayes factor indicates that the data slightly favor a medium or large positive effect over a negative or small positive one.

The one-sided hypothesis in the the left panel of Figure 8 assumes a rather flexible null hypothesis that can includes large negative effects as well as near equalities. In many cases, large negative effect sizes may be considered unreasonable or uninteresting *a priori*. If negative effects are unreasonable or uninteresting, it is possible to eliminate them from the test, because the Bayes factor will penalize the null hypothesis for the flexibility added by the range of negative values. The right panel of Figure 8 shows the researcher's hypothesis as the alternative against a hybrid null consisting of a mixture between a point nil at 0 and a region of unimportant positive effect sizes. This extension of the hybrid null allows the researcher to test their hypothesis against the hypothesis that the effect is unimportant and positive, or 0. This Bayes factor is reasonable when negative effects are unreasonable *a priori*.

To compute the hybrid one-sided Bayes factor, we again relax the assumption of a symmetric null region. First, we compute the hybrid Bayes factor of the nil versus all positive values of δ, by setting the null region to $(-\infty, 0)$ and $\pi_0 = 1$, and call this B_1. This yields the one-sided test of Wetzels et al. (2009). We then compute the Bayes NOH Bayes factor B_2, with null region $(0, c)$ and alternative region (c, ∞). The hybrid one-sided Bayes factor may then be computed by applying Eq. 11.

Applying the hybrid one-sided Bayes factor with $\pi_0 = 0.5$ and $\pi_0 = 1$ to Grider and Malberg's data yields Bayes factors of 1.91 for the null, and 1.7 for the alternative, respectively. Both of these Bayes factor values are concordant with the other Bayes factor values computed in this paper, suggesting that the data have little evidentiary value.

Discussion

In this paper we developed Bayes factors for testing equality and ordinal constraints. Our advocacy of Bayes factor is based on the concept of evidence, namely that researchers may consider it helpful to report the evidence in data for various positions than make decisions with specified error rates. The Bayes factor is the marginal probability of the data under two competing hypotheses, and this ratio is directly interpretable without recourse to decision regions or error rates. It is possible to use Bayes factors to compute quantities that are appropriate for decision-making, if desired. Posterior odds reflect an analyst's belief about the relative weighting of two hypotheses given observed data and are a thus a natural quantity for making decisions. We find both posterior odds and Bayes factor directly interpretable, though we prefer Bayes factor for scientific communication. Some researchers, however, may argue that posterior odds are more interpretable than Bayes factors. For example, there are Bayesian statistical equivalence tests based on posterior odds rather than Bayes factor (e.g., Selwyn, Dempster & Hall, 1981; Selwyn & Hall, 1984; Wellek, 2003). In fact, these tests are based on improper priors in which the prior odds and Bayes factors cannot be meaningfully defined. We view these approaches, in which one cannot quantify the amount of evidence contributed by the data, as less advantageous.

Researchers who prefer posterior odds to Bayes factors may still use our methods; they must, however, stipulate prior odds. Prior odds may be used to add context to an analysis; if a hypothesis is highly unlikely *a prior*, it may be assigned low prior odds. One example is Bem's (in press) recent claim that participants can sense random events before they occur, or precognition. We recently analyzed Bem's claims, and found that the Bayes factor computed from Dr. Bem's reported data was 40 in favor of ESP (Rouder and Morey, submitted). While this Bayes factor is large, we cautioned readers to use very unfavorable prior odds in interpreting the results because precognition is contrary to well-established laws and principles in physics and biology. If one begins with extremely low prior odds for a hypothesis, more evidence will be necessary. This strikes us as both reasonable and a reflection of how scientists reason; quantifying of this reasoning whenever possible will make the scientific process appropriately responsive to evidence. Prior odds are therefore useful for adding context to a Bayes factor; however, due to its insensitivity to prior odds, we recommend reporting of Bayes factor; computing posterior odds is then simply a matter of multiplication, as can easily be seen in Eq. 4.

Bayes factor, the ratio of marginal probabilities, is contingent on specification of a prior over parameters under competing hypotheses. We have presented a number of different specifications for nulls and alternatives. Researcher interested in these techniques my wonder which of the several null and alternatives they should use. Consider first the question of assessing equality constraints. Our recommendation is the hybrid setup in Figure 3D. This hybrid model is quite general and seems highly appropriate under many conditions. We provide an easy-to-use web applet that calculates hybrid Bayes factors at http://pcl.missouri.edu/bayesfactor, shown in Figure 9. For ordinal tests, we recommend either Bayes factor in Figure 8 for assessing ordinal constraints, depending on whether the researcher believes negative effects are reasonable *a priori*. Moreover, as previously discussed, in Bayesian statistics, there is no drawback to computing and reporting multiple tests.

Once a test is selected, researchers must also choose model parameters, such as equivalence regions or the weight parameter π_0. Choices of the equivalence regions and weights of the point nil reflect reasoned beliefs about the problem at hand. In fields where interesting effects are smaller (for example, subliminal priming), the width of the null region may be made correspondingly small. In other fields where interesting effect sizes are larger (for example, cognitive aging), the region may be made larger to suit. The task of selecting boundaries is simplified somewhat by the parameterization. The models are parameterized with respect to standardized effect size. General guidelines already exist (Cohen, 1988) and we note that many journals require reporting some measure of effect size. Certainly if researchers are able to interpret effect size measures in the context of existing literature, it is not difficult to extend this to setting bounds on equivalence regions. Eventually conventions may arise, as they have with type I error rate. However, because these Bayes factors are computed using only the t test statistic and sample size, any researcher may calculate a Bayes factor for themselves, using a different equivalence region.

In addition to specifying equivalence regions, computing the hybrid Bayes Factor requires specifying the mixture probability that null hypotheses are nil, π_0. The value of π_0 will depend on the type of test desired. For researchers who believe that nil hypotheses are impossible *a priori*, or who are uninterested in the nil, $\pi_0 = 0$ is a reasonable value. For researchers who want to test nil hypotheses against reasonable, nonnegligible values of δ, we recommend $\pi_0 = 1$ with an null interval including negligible effect sizes (Figure 7A). Other values may correspond to other models of interest. Setting π_0 should pose no great difficulty for researchers. If researchers report their test statistics, other researchers will be able to reproduce the analysis with different values, if desired. In all cases, however, the interpretation of Bayes factors should be understood in the context of the assumptions about the equivalence regions and π_0, and not as assumption-free measures of evidence.

It should also be noted that in spite of the wide range of models employed in this paper to compute Bayes factors, in the example using Grider and Malmberg's data, the Bayes factor ranged from about 2.7 for an effect, to 3.6 against, all indicating that there is little or no evidence for a substantial effect. It is not surprising that the Bayes factors vary, because the null hypothesis tested was in each case different: some were point hypothesis, some ranges of values, and some mixtures of the two. But the interpretation of the Bayes factor with respect to the hypothesis in question, whether we should take the experiment as evidence of an effect, is substantially the same.

Conclusion

We have presented here a Bayes factor approach to hypothesis testing that has the following advantages: first, it allows for null hypotheses that are not exactly nil. In this way, small, uninteresting effects are not emphasized. Second, the use of Bayes factors allows for the accumulation of evidence for both the null and the alternative hypotheses simultaneously. Third, because the Bayes factor is a relative measure, it does not overstate the evidence against a default hypothesis. Fourth, the framework suggested is sufficiently general to construct tests of equivalence and ordinal tests. The key innovation in these Bayes factors is that they allow researchers to accept a theoretically meaningful constraint even when it fails for trivial or uninteresting reasons, through the use of equivalence regions. This ability to judiciously quantify the evidence for and against constraints in real-world situations should lead to better understanding of lawfulness and parsimony in psychology.

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Appendix A

Derivation of Nonoverlapping Bayes Factor

The joint prior p_2 on δ and σ^2 for hypothesis H_2 is

$$p_2(\delta, \sigma^2) = \frac{1}{\pi_2 \sigma^2} t_{\nu_0,r}(\delta) I(\delta \in \Delta_2)$$

where $t_{\nu_0,r}$ is the density function of the central t distribution with ν_0 degrees of freedom and scale r, I is an indicator function, Δ_2 is the null region $(-c, c)$, and

$$\pi_2 = \int_{-c}^{c} t_{\nu_0,r}(\delta) d\delta$$

Similarly, the joint prior for H_3 is

$$p_3(\delta, \sigma^2) = \frac{1}{(1-\pi_2)\sigma^2} t_{\nu_0,r}(\delta) I(\delta \notin \Delta_2)$$

The NOH Bayes factor is the ratio of the marginal likelihoods,

$$B_{23} \;=\; \int_{\delta \in \Delta_2} \int_{\sigma^2} p(Y|\sigma^2, \delta) p_2(\delta, \sigma^2) d\delta \sigma^2 \quad \Big/ \int_{\delta \in \Delta_2} \int_{\sigma^2} p(Y|\sigma^2, \delta) p_3(\delta, \sigma^2) d\delta, d\sigma^2 \qquad (12)$$

To find a complete expression of the Bayes Factor, we first consider the joint posterior of δ and σ^2:

$$p(\delta, \sigma^2|Y) \;\propto\; \frac{1}{\pi_2} (\sigma^2)^{-\frac{N}{2}} \exp\left\{ -\frac{1}{2\sigma^2} \sum_i (y_i - \sigma\delta)^2 \right\} (\sigma^2)^{-1} \left(1 + \frac{(\delta/r)^2}{\nu_0} \right)^{-\frac{\nu_0+1}{2}} \qquad (13)$$

The constants with respect to σ^2 and δ can be safely ignored, because the will be the same for both the numerator and the denominator in Eq. 12. Expanding the square and distributing the sum yields

$$p(\delta, \sigma^2|y) \;\propto\; \frac{1}{\pi_2} (\sigma^2)^{-\frac{N}{2}} \exp\left\{ -\frac{1}{2\sigma^2} \left(\sum_i y_i^2 - 2\sigma\delta N\bar{Y} + N\sigma^2\delta^2 \right) \right\} (\sigma^2)^{-1} \left(1 + \frac{(\delta/r)^2}{\nu_0} \right)^{-\frac{\nu_0+1}{2}}$$

Using the identity $\sum_i y_i^2 = (N-1)s^2 + N\bar{y}^2$,

$$p(\delta, \sigma^2|y) \;\propto\; \frac{1}{\pi_2} (\sigma^2)^{-\frac{N}{2}} \exp\left\{ -\frac{1}{2\sigma^2} \left((N-1)s^2 + N\bar{Y}^2 - 2\sigma\delta N\bar{Y} + N\delta^2\sigma^2 \right) \right\}$$

$$
\begin{aligned}
& \times(\sigma^2)^{-1}\left(1+\frac{(\delta/r)^2}{\nu_0}\right)^{-\frac{\nu_0+1}{2}} \\
\propto \quad & (\sigma^2)^{-\frac{N}{2}}\exp\left\{-\frac{s^2}{2\sigma^2}\left((N-1)+\frac{N\bar{Y}^2}{s^2}-2\frac{\sigma\delta N\bar{Y}}{s^2}+\frac{N\sigma^2\delta^2}{s^2}\right)\right\} \\
& \times(\sigma^2)^{-1}\left(1+\frac{(\delta/r)^2}{\nu_0}\right)^{-\frac{\nu_0+1}{2}} \\
\propto \quad & \frac{1}{\pi_2}(\sigma^2)^{-\frac{N}{2}} \\
& \times\exp\left\{-\frac{1}{2\sigma^2/s^2}\left((N-1)+\frac{N\bar{Y}^2}{s^2}-2\sqrt{N}\delta\frac{\sigma}{s}\frac{\sqrt{N}\bar{Y}}{s}+N\delta^2\frac{\sigma^2}{s^2}\right)\right\} \\
& \times(\sigma^2)^{-1}\left(1+\frac{(\delta/r)^2}{\nu_0}\right)^{-\frac{\nu_0+1}{2}}
\end{aligned}
$$

We can substitute the following notation:

$$
\begin{aligned}
t &= \frac{\bar{Y}}{s/\sqrt{N}} \\
\lambda^2 &= \frac{\sigma^2}{s^2}
\end{aligned}
$$

The substitution yields

$$
\begin{aligned}
p(\delta,\sigma^2|y) \quad \propto \quad & \frac{1}{\pi_2}(\lambda^2)^{-\frac{N}{2}}(s^2)^{-\frac{N}{2}}\exp\left\{-\frac{N-1}{2\lambda^2}\right\} \\
& \times\exp\left\{-\frac{1}{2\lambda^2}\left(t^2-2\sqrt{N}\delta\lambda^2 t+N\delta^2\lambda^2\right)\right\} \\
& \times(\lambda^2)^{-1}(s^2)^{-1}\left(1+\frac{(\delta/r)^2}{\nu_0}\right)^{-\frac{\nu_0+1}{2}} \\
\propto \quad & \frac{1}{\pi_2}(\lambda^2)^{-\frac{N}{2}-1}\exp\left\{-\frac{N-1}{2\lambda^2}\right\} \\
& \times\exp\left\{-\frac{1}{2\lambda^2}\left(t-\delta\sqrt{N\lambda^2}\right)^2\right\} \\
& \times\left(1+\frac{(\delta/r)^2}{\nu_0}\right)^{-\frac{\nu_0+1}{2}}
\end{aligned}
$$

After a similar simplification for H_3, the NOH Bayes Factor is

$$
\mathrm{B}_{23} = \frac{\int_{\delta\in\Delta_2}\int_{\lambda^2}(\lambda^2)^{-\frac{N}{2}-1}\exp\left\{-\frac{N-1}{2\lambda^2}-\frac{1}{2\lambda^2}\left(t-\delta\sqrt{N\lambda^2}\right)^2\right\}\left(1+\frac{(\delta/r)^2}{\nu_0}\right)^{-\frac{\nu_0+1}{2}}d\lambda^2 d\delta}{\int_{\delta\in\Delta_3}\int_{\lambda^2}(\lambda^2)^{-\frac{N}{2}-1}\exp\left\{-\frac{N-1}{2\lambda^2}-\frac{1}{2\lambda^2}\left(t-\delta\sqrt{N\lambda^2}\right)^2\right\}\left(1+\frac{(\delta/r)^2}{\nu_0}\right)^{-\frac{\nu_0+1}{2}}d\lambda^2 d\delta}\times\frac{1-\pi_2}{\pi_2}
$$

where Δ_3 is the complement of Δ_2.

Appendix B

Approaches to computing the Bayes factor

One standard approach to numerical integration is Gaussian quadrature (Abramowitz & Stegun, 1965; Press, Flannery, Teukolsky, & Vetterling, 1988), which is implemented in many software packages including Matlab and R. We have found that Gaussian quadrature integration is very quick in practice, but can be unstable in circumstances where the posterior mass is highly concentrated over a small interval.

A second approach is using Normal approximations to the marginal posterior distribution of δ. As before, Gaussian quadrature may be used to integrate out σ^2, yielding the marginal posterior on δ. For reasonable sample sizes, the marginal posterior will be approximately Normally distributed (Bernardo & Smith, 2000). Laplace's method (Laplace, 1774/1986) approximates an integral using the normal distribution function, leading to a highly accurate approximation for the integral and thus the posterior odds. The prior odds may be computed easily by standard statistical software. In cases where Gaussian quadrature fails, Laplace's method may be used to compute the NOH Bayes factor quickly and accurately.

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Footnotes

[1]A point hypothesis on a parameter takes the form $\theta = c$ The hypothesis may be rewritten as $\theta' = 0$, where $\theta' = \theta - c$.

[2]The distribution function of normal distribution with mean μ and variance σ^2 is

$$\phi(y \mid \mu, \sigma^2) = (2\pi\sigma^2)^{-\frac{1}{2}\exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}}$$

[3]Consistency can be obtained in conventional NHST if the type I error rate is reduced to zero with increasing sample size, but this is not done in practice and to our knowledge no one has proposed a way of doing so.

Figure Captions

Figure 1. Statistical equivalence testing with confidence intervals. The shaded region represents the equivalence region. When the confidence interval is completely within the equivalence region, the null hypothesis of nonequivalence is rejected.

Figure 2. Evidence against the null hypothesis given a just-significant p value. The square represents the likelihood of a t value of 1.75 under the nil hypothesis, which is here represented by a central t distribution (dark density). The circle represents the likelihood of a t value of 1.75 given the alternative that the true standardized effect size is 0.2, represented here by a noncentral t distribution (lighter density).

Figure 3. Four models to be considered for the one-sample t test Bayes Factor. A: The null hypothesis is a point, and the alternative is a Cauchy distribution. B: The null and alternative models are both Cauchy distributions, with different scales. C: The null and alternative are different intervals from the same Cauchy distribution. D: The null hypothesis includes both an interval and a point-null component.

Figure 4. Bayes factor as a function of sample size for a true effect size of $\delta = 1$. Plotted are the Bayes factor for the median t value at each sample size. The overlapping-hypotheses Bayes Factor does not converge to 0, while the JZS Bayes Factor and the nonoverlapping Bayes Factor both converge.

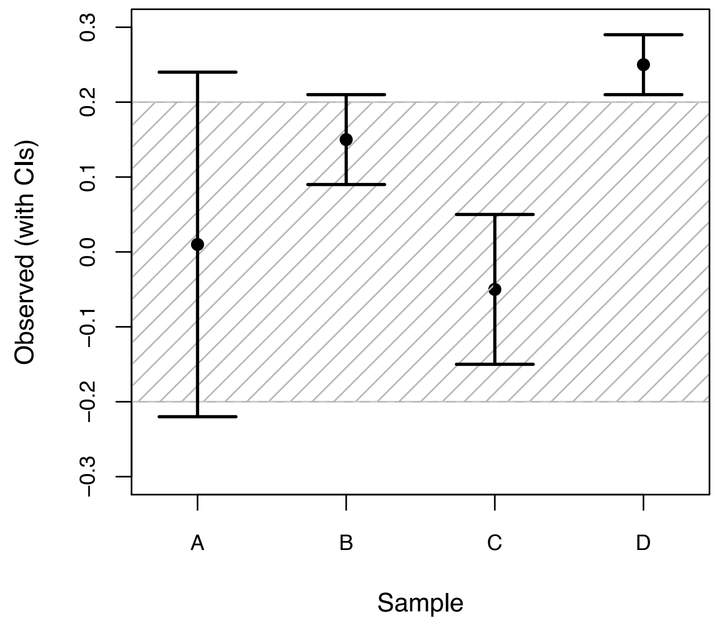
Figure 5. Nonoverlapping Bayes factor as a function of sample size for a few true effect sizes. Plotted are the Bayes factor for the median t value at each sample size. The solid, dashed, and dashed-dotted lines represent true effect sizes of $\delta = 0, 0.2, 0.5$. For each line type, the upper and middle line are the NOH Bayes factor with the Cauchy and Normal prior, and the lower line is the corresponding JZS Bayes factor. The null interval is (-0.2,0.2).

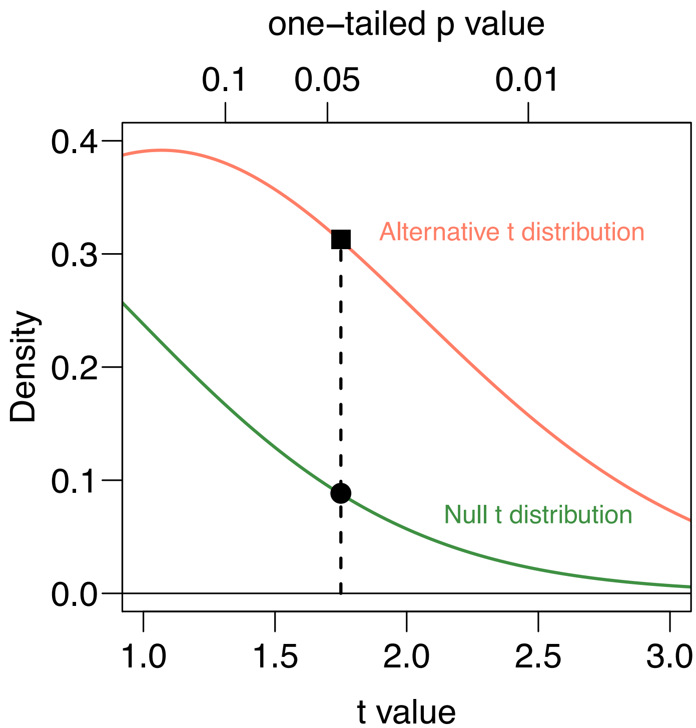
Figure 6. Hybrid Bayes factor as a function of sample size for a few true effect sizes. Plotted are the Bayes factor for the median t value at each sample size. The solid, dashed, and dashed-dotted lines represent true effect sizes of $\delta = 0, 0.2, 0.5$. For each line type, the upper and middle line are the NOH Bayes factor with the Cauchy and Normal prior, and the lower line is the corresponding JZS Bayes factor. The null hypothesis has mixing probability $\pi_0 = 0.5$ and extends on the interval $(-.2, .2)$.

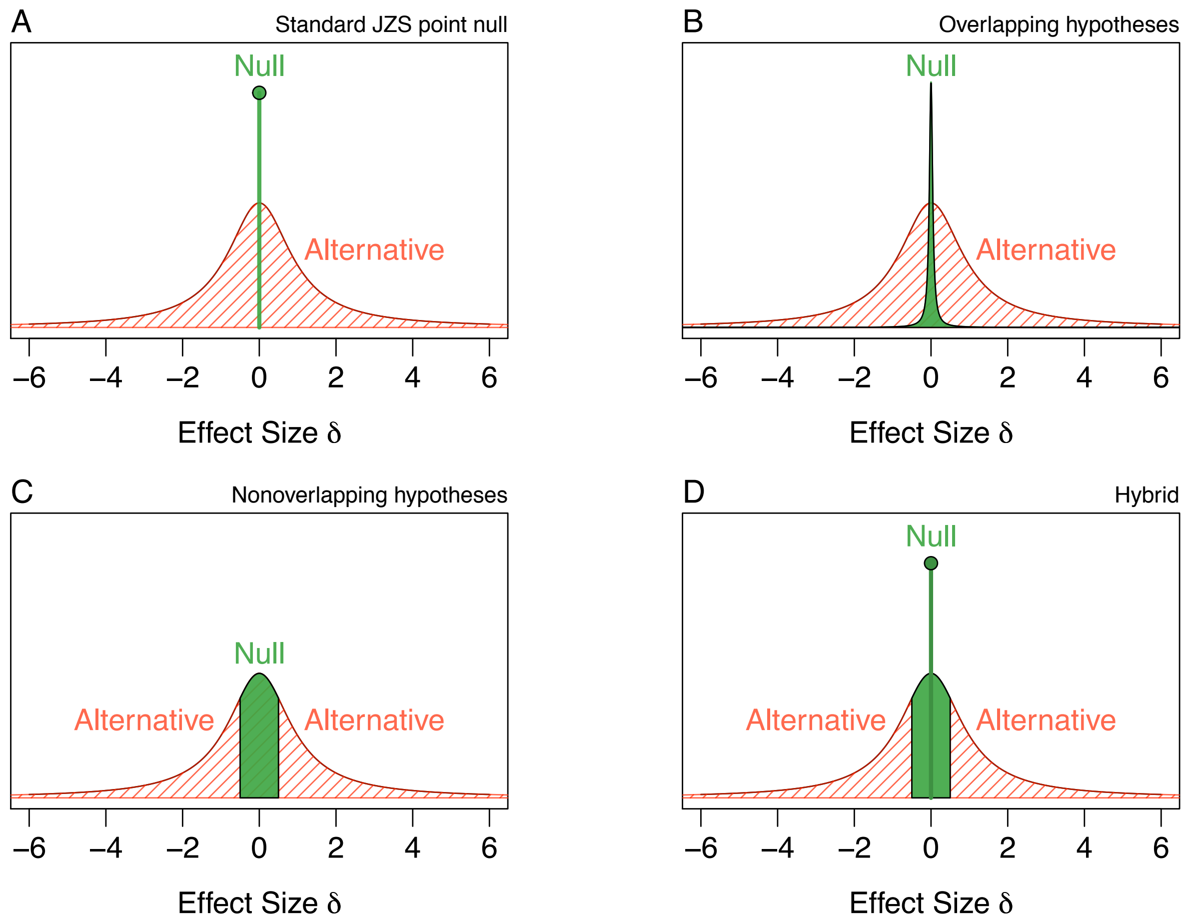
Figure 7. A: The hybrid model with $\pi_0 = 1$. There is an interval of small effect sizes which cannot occur under either hypothesis. B: Bayes factor as a function of sample size for a few true effect sizes. Plotted are the Bayes factor for the median t value at each sample size. The solid, dashed, and dashed-dotted lines represent true effect sizes of $\delta = 0, 0.2, 0.5$. For each line type, the upper and middle line are the NOH Bayes factor with the Cauchy and Normal prior, and the lower line is the corresponding JZS Bayes factor. The null region was (-0.2,0.2) and $\pi_0 = 1$.

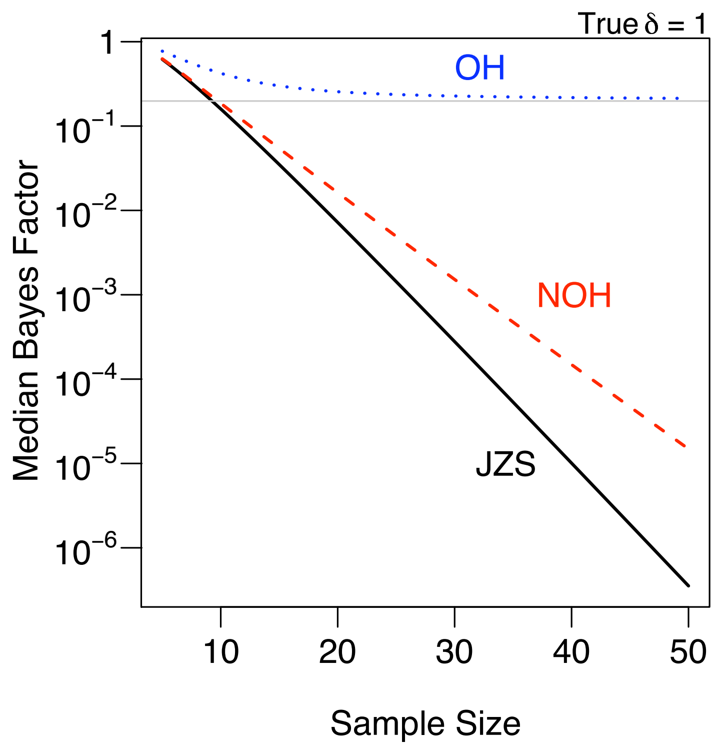
Figure 8. Possible one-sided hypotheses. The left panel shows hypotheses for a test of important positive effects against all other effects. The right panel shows hypotheses for a test of important positive effects against either no effect or unimportant positive effects.

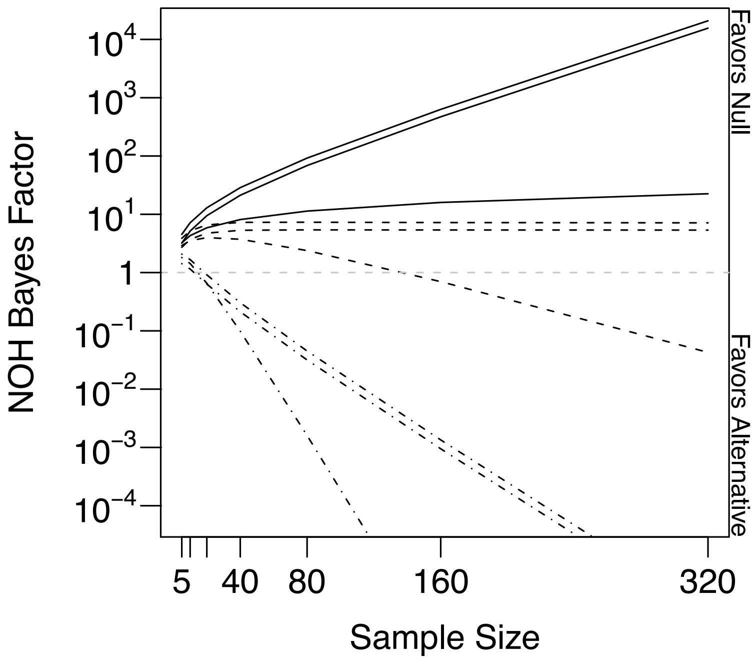
Figure 9. The web applet at http://pcl.missouri.edu/bayesfactor for computing area and hybrid Bayes factors. Users specify the sample size, t value, equivalence region, π_0, and the prior scale. The web applet returns the corresponding Bayes factor.

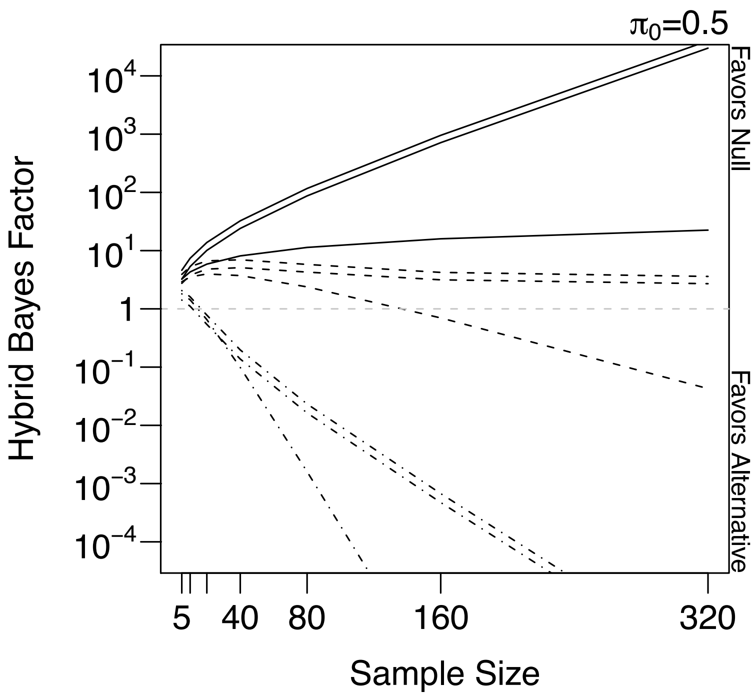

0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
Observed (with CIs)
A
B
C
D
Sample

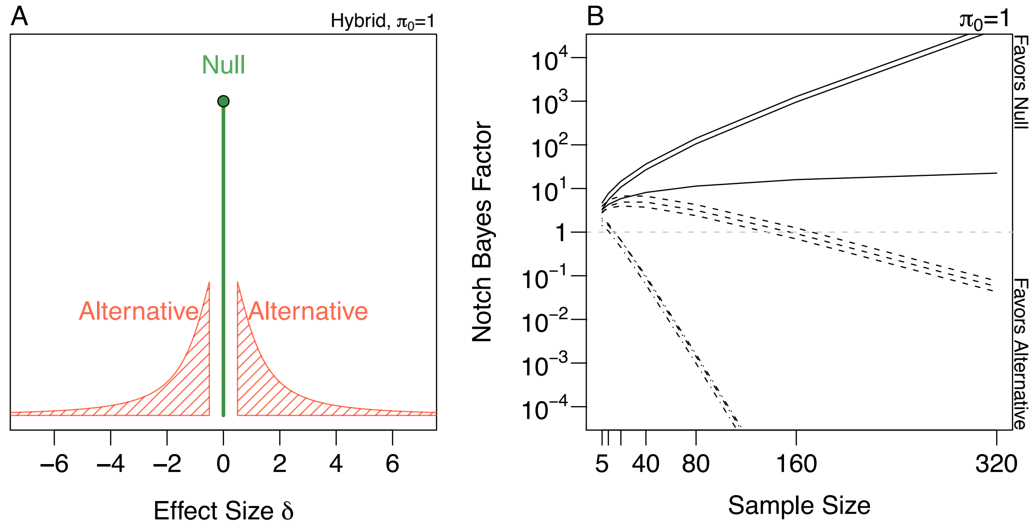

one−tailed p value
0.1
0.05
0.01
0.4
0.3
0.2
0.1
0.0
Density
Alternative t distribution
Null t distribution
1.0
1.5
2.0
2.5
3.0
t value


A
Standard JZS point null
Null
Alternative
−6
−4
−2
0
2
4
6
Effect Size δ
B
Overlapping hypotheses
Null
Alternative
Effect Size δ
C
Nonoverlapping hypotheses
Null
Alternative
Alternative
Effect Size δ
D
Hybrid
Null
Alternative
Alternative
Effect Size δ

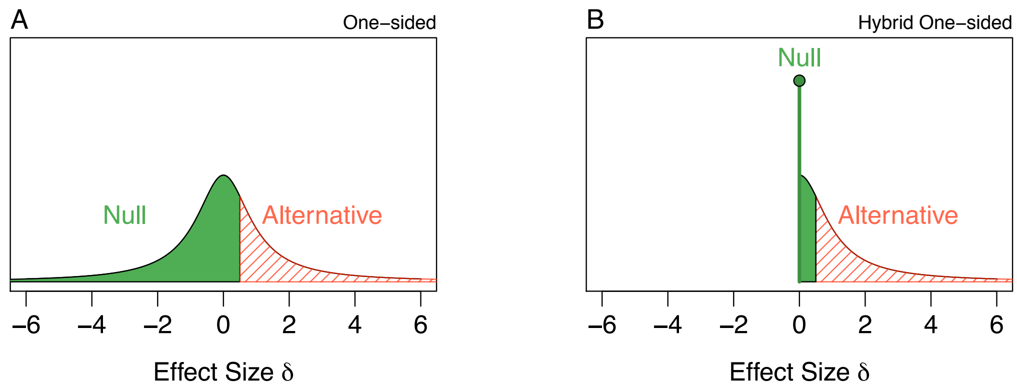

True δ = 1
OH
NOH
JZS
Median Bayes Factor
1
10^−1
10^−2
10^−3
10^−4
10^−5
10^−6
10
20
30
40
50
Sample Size

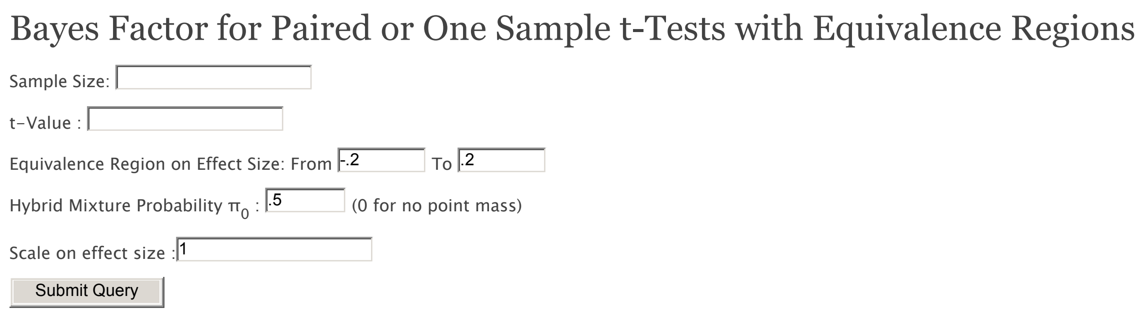

NOH Bayes Factor
10^4
10^3
10^2
10^1
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
Favors Null
Favors Alternative
5
40
80
160
320
Sample Size


π0=0.5
Hybrid Bayes Factor
10^4
10^3
10^2
10^1
1
10^-1
10^-2
10^-3
10^-4
Favors Null
Favors Alternative
5
40
80
160
320
Sample Size



A
Hybrid, π_0=1
Null
Alternative
Alternative
−6
−4
−2
0
2
4
6
Effect Size δ
B
π_0=1
10^4
10^3
10^2
10^1
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
Notch Bayes Factor
Favors Null
Favors Alternative
5
40
80
160
320
Sample Size


A
One−sided
Null
Alternative
−6
−4
−2
0
2
4
6
Effect Size δ
B
Hybrid One−sided
Null
Alternative
−6
−4
−2
0
2
4
6
Effect Size δ



Bayes Factor for Paired or One Sample t-Tests with Equivalence Regions

Sample Size:

t–Value :

Equivalence Region on Effect Size: From -.2 To .2

Hybrid Mixture Probability π_0 : .5 (0 for no point mass)

Scale on effect size : 1

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