1 Model Specification

Let y_{ij} be the jth replicate observation for the ith participant, $i = 1, \ldots, I$, $j = 1, \ldots, J$. The model of data is

$$y_{ij}|\mu_i, \sigma^2 \overset{iid}{\sim} \text{Normal}(\mu_i, \sigma^2). \tag{1}$$

2 Prior Specification

We place a hyperprior on parameters as follows:

2.1 First Level

The prior on σ^2 is

$$\sigma^2 \sim \text{Inverse Gamma}(a, b), \tag{2}$$

where a and b serve as *prior settings* that must be specified before analysis without recourse to the data.

We place multilevel prior on μ_i as follows:

$$\mu_i|\theta, \delta \overset{iid}{\sim} \text{Normal}(\theta, \delta). \tag{3}$$

2.2 Second Level

We place the following priors on group mean θ and group variance δ:

$$\theta \sim \text{Normal}(c, d), \tag{4}$$
$$\delta \sim \text{Inverse Gamma}(e, d), \tag{5}$$

where c, d, e, and f are prior settings that must be specified *a priori*.

3 Development

I break the development into two steps: 1. deriving the joint posterior, 2. deriving conditional posterior for each parameter.

3.1 Joint Posterior Distribution

Let $\boldsymbol{y}$ be the matrix of responses, and let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_I)$ be the vector participant means. Bayes rule implies

$$f(\boldsymbol{\mu}, \sigma^2, \theta, \delta|\boldsymbol{y}) \propto f(\boldsymbol{y}|\boldsymbol{\mu}, \sigma^2, \theta, \delta) f(\boldsymbol{\mu}, \sigma^2, \theta, \delta). \tag{6}$$

Let's first look at the likelihood term $f(\boldsymbol{y}|\boldsymbol{\mu}, \sigma^2, \theta, \delta)$. Multiplication yields

$$f(\boldsymbol{y}|\boldsymbol{\mu}, \sigma^2, \theta, \delta) \propto \frac{1}{(\sigma^2)^{IJ/2}} \exp\left(-\frac{\sum_i \sum_j (y_{ij} - \mu_i)^2}{2\sigma^2}\right). \tag{7}$$

Now, let's look at the prior term, $f(\boldsymbol{\mu}, \sigma^2, \theta, \delta)$. This is a joint prior, so by the definition of conditional probability, it may be represented as the product of a conditional and a marginal:

$$\begin{aligned} f(\boldsymbol{\mu}, \sigma^2, \theta, \delta) &= f(\boldsymbol{\mu}, \sigma^2|\theta, \delta) f(\theta, \delta) = f(\boldsymbol{\mu}|\theta, \delta) f(\sigma^2) f(\theta) f(\delta) & (8) \\ &\propto \frac{1}{\delta^{I/2}} \exp\left(-\frac{\sum_i (\mu_i - \theta)^2}{2\delta}\right) \frac{1}{(\sigma^2)^{a+1}} \exp(-b/\sigma^2) \times \\ &\quad \exp\left(-\frac{(\theta - c)^2}{d}\right) \frac{1}{\delta^{e+1}} \exp(-f/\delta) & (9) \end{aligned}$$

The joint posterior is proportional to the product of (7) and (9):

$$\begin{aligned} f(\boldsymbol{\mu}, \sigma^2, \theta, \delta|\boldsymbol{y}) &\propto \frac{1}{(\sigma^2)^{IJ/2}} \exp\left(-\frac{\sum_i \sum_j (y_{ij} - \mu_i)^2}{2\sigma^2}\right) \times \\ &\quad \frac{1}{\delta^{I/2}} \exp\left(-\frac{\sum_i (\mu_i - \theta)^2}{2\delta}\right) \frac{1}{(\sigma^2)^{a+1}} \exp(-b/\sigma^2) \times \\ &\quad \exp\left(-\frac{(\theta - c)^2}{d}\right) \frac{1}{\delta^{e+1}} \exp(-f/\delta) & (10) \end{aligned}$$

3.2 Conditional Posterior Distributions

Let $\boldsymbol{\mu}_{-i}$ be all parameters in $\boldsymbol{\mu}$ except μ_i, i.e., $\boldsymbol{\mu}_{-1} = (\mu_2, \ldots, \mu_I)$.

3.2.1 μ_i

Placing all terms that do not depend on a specific μ_i into the constant of proportionality yields

$$f(\mu_i|\boldsymbol{\mu}_{-i}, \sigma^2, \theta, \delta, \boldsymbol{y}) \propto \exp\left(-\frac{\sum_j (y_{ij} - \mu_i)^2}{2\sigma^2}\right) \exp\left(-\frac{(\mu_i - \theta)^2}{2\delta}\right) \quad (11)$$

Equation (11) is recognizable, and completing the square (Rouder & Lu, 2005, p.xx) yields

$$\mu_i|\boldsymbol{\mu}_{-i}, \sigma^2, \theta, \delta, \boldsymbol{y} \sim \text{Normal}(k_i v, v) \quad (12)$$

where

$$\begin{aligned} v &= \left(\frac{J}{\sigma^2} + \frac{1}{\delta}\right)^{-1} \\ k_i &= \left(\frac{J\bar{y}_i}{\sigma^2} + \frac{\theta}{\delta}\right), \end{aligned}$$

where $\bar{y}_i$ is the sample mean for the ith subject.

3.2.2 σ^2

Placing all terms that do not depend on σ^2 into the constant of proportionality yields

$$f(\sigma^2|\boldsymbol{\mu},\theta,\delta,\boldsymbol{y}) \propto \frac{1}{(\sigma^2)^{IJ/2}} \exp\left(-\frac{\sum_i \sum_j (y_{ij}-\mu_i)^2}{2\sigma^2}\right) \frac{1}{(\sigma^2)^{a+1}} \exp(-b/\sigma^2) \tag{13}$$

Equation (13) is recognizable, and is analyzed in Rouder & Lu, 2005, p.xx. The conditional posterior may be expressed as

$$\sigma^2|\boldsymbol{\mu},\theta,\delta,\boldsymbol{y} \sim \text{Inverse Gamma}\left(a+IJ/2,\;\; b+\sum_j \sum_i (y_{ij}-\mu_i)^2/2\right). \tag{14}$$

3.2.3 θ

Placing all terms that do not depend on θ into the constant of proportionality yields

$$f(\theta|\boldsymbol{\mu},\sigma^2,\delta,\boldsymbol{y}) \propto \exp\left(-\frac{\sum_i (\mu_i-\theta)^2}{2\delta}\right) \exp\left(-\frac{(\theta-c)^2}{2d}\right). \tag{15}$$

Equation (15) is recognizable, and completeing the square (Rouder & Lu, 2005, p.xx) yields

$$\theta|\boldsymbol{\mu},\sigma^2,\delta,\boldsymbol{y} \sim \text{Normal}(k^*v^*, v^*) \tag{16}$$

where

$$\begin{aligned} v^* &= \left(\frac{I}{\delta}+\frac{1}{d}\right)^{-1} \\ k^* &= \left(\frac{I\bar{\mu}}{\delta}+\frac{c}{d}\right), \end{aligned}$$

where $\bar{\mu} = (\sum_i \mu_i)/I$.

3.2.4 δ

Placing all terms that do not depend on δ into the constant of proportionality yields

$$f(\delta|\boldsymbol{\mu},\sigma^2\theta,\boldsymbol{y}) \propto \frac{1}{\delta^{I/2}} \exp\left(-\frac{\sum_i (\mu_i-\theta)^2}{2\delta}\right) \frac{1}{\delta^{e+1}} \exp(-f/\delta) \tag{17}$$

Equation (13) is recognizable, and is analyzed in Rouder & Lu, 2005, p.xx. The conditional posterior may be expressed as

$$\delta|\boldsymbol{\mu},\sigma^2,\theta,\boldsymbol{y} \sim \text{Inverse Gamma}\left(e+I/2,\;\; f+\sum_i (\mu_i-\theta)^2/2\right). \tag{18}$$